Representation Theory — Exercise Sheet 10 | **TU Kaiserslautern**
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Throughout G and H denote finite groups.

Exercise 41.
Let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$. Then $|G| = \sum_{i=1}^{r} \chi_i(1)^2$.

Exercise 42.
Define a map

$$\begin{aligned}\langle\ ,\ \rangle : \mathrm{Cl}(G) \times \mathrm{Cl}(G) &\to \mathbb{C}\\ (\chi, \psi) &\mapsto \langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1})\end{aligned}$$

Prove the following.

(a) $\langle\ ,\ \rangle$ is bilinear, symmetric and positive definite.

(b) $\mathrm{Irr}(G)$ is an orthonormal basis for $\mathrm{Cl}(G)$.

Exercise 43.
Let $X_1 : G \to \mathrm{GL}_n(\mathbb{C})$ and $X_2 : H \to \mathrm{GL}_m(\mathbb{C})$ be complex representations for some $n, m \geq 1$, with characters χ_1 and χ_2 respectively. For $(g, h) \in G \times H$, let $X_1(g) = (a_{ij})_{1 \leq i \leq n, 1 \leq j \leq n}$ and define

$$(X_1 \otimes X_2)(g, h) := \begin{pmatrix} a_{11}(g)X_2(h) & \dots & a_{1n}(g)X_2(h) \\ \vdots & & \vdots \\ a_{n1}(g)X_2(h) & \dots & a_{nn}(g)X_2(h) \end{pmatrix}.$$

(a) Prove that the map

$$\begin{aligned}X_1 \otimes X_2 : G \times H &\to \mathrm{GL}_{nm}(\mathbb{C})\\ (g, h) &\mapsto (X_1 \otimes X_2)(g, h)\end{aligned}$$

is a matrix representation of $G \times H$ with character $\chi = \chi_1\chi_2$ where

$$\chi(g, h) := \chi_1(g)\chi_2(h)$$

for all $(g, h) \in G \times H$.

(b) Prove that

$$\mathrm{Irr}(G \times H) = \{\theta\psi \mid \theta \in \mathrm{Irr}(G), \psi \in \mathrm{Irr}(H)\}.$$

Exercise 44.
Let $\chi, \lambda \in \mathrm{Irr}(G)$ and suppose that λ is linear. Show that $\chi\lambda \in \mathrm{Irr}(G)$.