

Throughout this exercise sheet K denotes a field of arbitrary characteristic and G a finite group. Each Exercise is worth 4 points.

EXERCISE 1

Let V be a finite-dimensional K -vector space. Prove the following assertions:

- (a) If $\rho : G \rightarrow \text{GL}(V)$ is a K -representation of G , then V becomes a G -set via the left action

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto g.v := \rho(g)(v) \end{aligned}$$

and moreover the following conditions are satisfied $\forall g \in G, \forall x, y \in V$, and $\forall \lambda \in K$:

- (i) $g.(x + y) = g.x + g.y$; and
(ii) $g.(\lambda x) = \lambda(g.x)$.
- (b) Conversely a linear action of G on V , i.e. an action $G \times V \rightarrow V$ satisfying (i) and (ii), gives rise to a K -representation $\rho : G \rightarrow \text{GL}(V)$ by setting $\rho(g)(v) := g.v \forall g \in G, \forall v \in V$.

EXERCISE 2

Assume $\text{char}(K) \neq 2$ and let $G := C_2 \times C_2$ be the Klein-four group. Find four pairwise non-equivalent (matrix) representations of G over K of degree one.

EXERCISE 3

- (a) Prove that the trivial representation of G is a sub-representation of any permutation representation of G .
- (b) Assume $K = \mathbb{C}$ and $G = C_3$. Find a decomposition into a direct sum of three irreducible representations of the regular representation of G .
[Hint: use (a) and $\omega \in \mathbb{C}$ a cubic root of unity to find three G -invariant subspaces of $V := \langle \{e_g \mid g \in G\} \rangle_K$]

EXERCISE 4 (Maschke's Theorem does not hold without the assumption that $\text{char}(K) \nmid |G|$.) Let p be a prime number, let $G := C_p = \langle g \mid g^p = 1 \rangle$ and let $K := \mathbb{F}_p$. Let $B := (e_1, e_2)$ be the standard ordered basis of $V := K^2$. Consider the matrix representation

$$\begin{aligned} R: G &\longrightarrow \text{GL}_2(K) \\ g^b &\longmapsto \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \end{aligned}$$

- (a) Prove that $W := Ke_1$ is G -invariant and deduce that R is reducible.
- (b) Prove that there is no direct sum decomposition of V into irreducible G -invariant subspaces.