Throughout this exercise sheet unless otherwise specified, K denotes a field of arbitrary characteristic, G a finite group, V a K-vector space and all vector spaces are assumed to be finite-dimensional. Recall that a representation is called *faithful* if it is injective.
Each exercise is worth 4 points.

Exercise 1
Let $N \trianglelefteq G$ and let $\pi : G \longrightarrow G/N$ be the quotient homomorphism. Given a K-representation $\rho : G/N \longrightarrow \mathrm{GL}(V)$, we set

$$\mathrm{Inf}_{G/N}^{G}(\rho) := \rho \circ \pi : G \longrightarrow \mathrm{GL}(V)\,,$$

the **inflation of** ρ **from** G/N **to** G. This is a K-representation of G.

(a) Prove that if ρ is irreducible, then $\mathrm{Inf}_{G/N}^{G}(\rho)$ is also irreducible.

(b) Assume that ρ is faithful. Compute the kernel of $\mathrm{Inf}_{G/N}^{G}(\rho)$.

Exercise 2
Let $G := S_n$ be the symmetric group on n letters ($n \in \mathbb{Z}_{\geq 1}$).

(a) Describe all the irreducible $\mathbb{C}$-representations of the symmetric group S_3.

(b) Exhibit two irreducible $\mathbb{C}$-representation of degree 1 and one irreducible $\mathbb{C}$-representation of degree 2 of S_4.

(c) Prove that S_n has at most two irreducible K-representations of degree 1. Under what condition on K does S_n have exactly two irreducible K-representations of degree 1?

Exercise 3 (Alternative proof of Maschke's Theorem over the field $\mathbb{C}$)
Let V be a G-vector space over $\mathbb{C}$.

(a) Prove that there exists a G-invariant scalar product $\langle -, - \rangle : V \times V \longrightarrow \mathbb{C}$, i.e. such that

$$\langle g.u, g.v \rangle \;=\; \langle u, v \rangle \quad \forall\, g \in G, \forall\, u, v \in V\,.$$

[Hint: consider a not-necessarily G-invariant scalar product on V, and use a sum on the elements of G to produce a G-invariant one.]

(b) Deduce that every G-invariant subspace W of V admits a G-invariant complement.

Exercise 4
The quaternion group Q_8 is the group containing 8 elements $\{1, -1, i, -i, j, -j, k, -k\}$ with the following multiplicative rules:

$$(-1)^2 = 1; \quad (-1)x = x(-1) = -x \;\forall x \in \{i, j, k\}; \quad i^2 = j^2 = k^2 = ijk = -1. \tag{1}$$

Let $I = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}$ be elements of $\mathrm{GL}_2(\mathbb{C})$ where $\mathbf{i} := \sqrt{-1} \in \mathbb{C}$.

(a) Show that $Q_8 \cong \langle I, J, K \rangle$.

(b) Deduce that Q_8 has an irreducible faithful $\mathbb{C}$-representation of degree two.