Prof. Gunter Malle **Dr. Niamh Farrell**

Due date: **Wednesday 27.06.18, 14:00** SS 2018

Note: Contrary to what I said in the last exercise class, we will not have any extra exercise classes scheduled this semester. Instead, this exercise sheet and the next one will carry 24 points each.

Throughout this exercise sheet V denotes a finite dimensional $\mathbb{C}$-vector space.

Exercise 1 (4 points)
Prove that the character table of a finite group G can be determined from its class multiplication constants m_{jkl}.
[Hint: For $k = 1, \dots, h$ let $M_k = \left(m_{jkl}\right)_{jl}$ be the matrix with jl-entry m_{jkl}. Set $w_i := (w_i(\hat{C}_1), \dots, w_i(\hat{C}_h))^t$. First show that $M_k w_i = w_i(\hat{C}_k) w_i$, and hence deduce that the M_k are simultaneously diagonalizable.]

Exercise 2 (4 points)
Let $G = A_4$.

(a) Determine the derived subgroup G' of G.

(b) Find the character table of G.

Exercise 3 (4 points)
Let G be a finite group. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible $\mathbb{C}$-representation of G of degree n affording the character χ. Prove the following two statements.

(a) If $z \in Z(G)$ then $\chi(z) = n\zeta$ for some $o(z)$-th root of unity ζ.

(b) If ρ is faithful and $g \in G$, then: $g \in Z(G)$ if and only if $|\chi(g)| = \chi(1)$.

Exercise 4 (4 points)
Let G be a finite group, let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_r\}$ and let

$$\chi = \sum_{i=1}^{r} n_i \chi_i$$

be a character of G for some $n_i \geq 0$.

(a) (1 point) Prove that $|\chi(g)| \leq \chi(1)$ for all $g \in G$.

(b) (1 point) Show that $\ker \chi = \bigcap_{i=1}^{r} \{\ker \chi_i \mid n_i > 0\}$.

(c) (2 points) Suppose that χ is faithful. Let $H \leq G$, let $\mathrm{Irr}(H) = \{\lambda_1, \dots, \lambda_s\}$ and suppose that

$$\chi|_H = \sum_{j=1}^{s} m_j \lambda_j$$

for some $m_j \geq 0$. Prove that H is abelian if and only if λ_j is linear for each $1 \leq j \leq s$ such that $m_j > 0$.

Exercise 5 (8 points)
Let G be a finite group and let $\Delta : G \to \mathrm{GL}(V)$ be an irreducible representation of G affording a character $\chi \in \mathrm{Irr}(G)$. Prove that

$$\chi(1) \mid |G : Z(\chi)|.$$

The following steps may be a helpful guide.

(a) (2 points) Show that it is enough to consider the case where $\ker \chi = 1$, and that if this holds then $Z(\chi) = Z(G)$.

(b) (2 points) Assume part (a). Let $G^m := G \times \cdots \times G$ (m times) and let $\Delta_m := \Delta \times \cdots \times \Delta$ (m times) be an irreducible representation of G^m. Let

$$H := \{(z_1, \dots, z_m) \in Z(G)^m \mid z_1 \dots z_m = 1\} \trianglelefteq G^m$$

and note that $|H| = |Z(G)|^{m-1}$.

- What is the irreducible character χ^m of G^m corresponding to Δ_m?
- Show that χ^m can be uniquely identified with an irreducible character $\overline{\chi^m}$ of G^m/H.

(c) (4 points) Assume parts (a) and (b). Prove that $\chi(1) \mid |G : Z(\chi)|$.
[Hint: Let $\alpha = \frac{\chi(1)}{\gcd(\chi(1), |G:Z(\chi)|)}$ and show that $\alpha = 1$ by finding an upper bound for α^m for any $m \geq 1$.]