Character Theory of Finite Groups — Exercise Sheet 5 TU Kaiserslautern

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Throughout this exercise sheet G denotes a finite group and V denotes a finite dimensional $\mathbb{C}$-vector space.

Exercise 1 (4 points)
Let $N \trianglelefteq G$ and let $\chi \in \mathrm{Irr}\,(G)$. Let ρ_{reg} denote the regular character of G/N. Prove that

$$(\mathrm{Res}^G_N(\chi))^G = \mathrm{Inf}^G_{G/N}(\rho_{\mathrm{reg}})\chi.$$

Exercise 2 (10 points)
Let ρ be a character of G and let $D : G \to \mathrm{GL}(V)$ be a $\mathbb{C}$-representation affording ρ. Let $\det : \mathrm{GL}(V) \to \mathbb{C}^*$ denote the determinant homomorphism. Define

$$\det \rho := \det \circ D : G \to \mathbb{C}^*$$

(a) (2 points) Show that $\det \rho$ does not depend on the choice of D and that it is a linear character of G.

(b) (4 points) Prove that if G is a simple group and $\chi \in \mathrm{Irr}(G)$ then $\chi(1) \neq 2$.

(c) (4 points) Let $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab^{-1} = a^{-1}\rangle$ be the dihedral group of order 8 and let Q_8 be the quaternion group of order 8 as described on Sheet 2, Exercise 4. Recall that D_8 and Q_8 have the same character table. Prove that D_8 and Q_8 can be distinguished by the determinants of their irreducible $\mathbb{C}$-representations.
[Hint: Let χ and ψ denote the irreducible characters of degree two of D_8 and Q_8 respectively. Show that $\det \chi \neq 1_{D_8}$ and $\det \psi = 1_{Q_8}$.]

Exercise 3 (6 points)
Prove the following.

(a) (2 points) Let $H, K \leq G$ with $HK = G$ and suppose that φ is a class function of H. Show that $(\varphi^G)_K = (\varphi_{H\cap K})^K$.

(b) (4 points) Let $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$. Prove that $\theta^G \in \mathrm{Irr}(G)$ if and only if $I_G(\theta) = N$.

Exercise 4 (4 points)
Let χ be a character of finite group G such that $|G| = m$ (for some $m \in \mathbb{N}$). Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_m/\mathbb{Q})$. Let χ^σ be given by $\chi^\sigma(g) := \chi(g)^\sigma$ for all $g \in G$. Prove that χ^σ is a character of G, and show that if χ is irreducible then so is χ^σ.