Rationality of blocks of quasi-simple finite groups

Niamh Farrell

City, University of London

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1. Introduction

ℓ a prime
$k = \overline{\mathbb{F}}_\ell$ an algebraic closure of the field of ℓ elements

Suppose P is a given finite ℓ-group
Let $\mathbb{B}_P := \{$ blocks of kG with defect group isomorphic to $P | G$ a finite gp$\}$

Donovan's Conjecture

$\mathbb{B}_P$ has finitely many Morita equivalence classes of blocks.

$\Updownarrow$ [Kessar, 2004]

Weak Donovan's Conjecture

The entries of the Cartan matrices of blocks in $\mathbb{B}_P$ are bounded by a function depending only on $|P|$.

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Rationality Conjecture

The Morita Frobenius numbers of blocks in $\mathbb{B}_P$ are bounded by a function depending only on $|P|$.

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1. Introduction

A a finite dimensional k-algebra, $m \in \mathbb{N}$

Definition

The *m-th Frobenius twist* of A, $A^{(\ell^m)}$, is

- a finite dimensional k-algebra with
- the same ring structure as A
- and twisted scalar multiplication

$$\lambda.x = \lambda^{\frac{1}{\ell^m}} x \quad \forall\, x \in A, \lambda \in k.$$

- $A \cong A^{(\ell^m)}$ as rings
- A and $A^{(\ell^m)}$ may not even be Morita equivalent as k-algebras

Definition

The *Morita Frobenius number* of a finite dimensional k-algebra A is the least $m \in \mathbb{N}$ such that A is Morita equivalent to $A^{(\ell^m)}$, denoted by $mf(A)$.

1. Introduction

Theorem (F.)

Let B be an ℓ-block of a quasi-simple finite group G and let $\bar{G} = G/Z(G)$. Suppose that one of the following holds.

(a) *$\bar{G}$ is an alternating group*

(b) *$\bar{G}$ is a sporadic group*

(c) *$\bar{G}$ is a finite group of Lie type in characteristic ℓ*

Then $mf(B) = 1$.

2. Blocks of finite groups of Lie type in non-defining char.

$\mathbf{G}$ a simple, simply-connected algebraic group defined over $\overline{\mathbb{F}}_p$ where $p \neq \ell$
$F : \mathbf{G} \to \mathbf{G}$ Frobenius morphism wrt an $\mathbb{F}_q$-structure, $q = p^a$, $a \geq 1$
$\mathbf{G}^F$ the finite group of fixed points

I. "Filing system" for blocks of $k\mathbf{G}^F$

[Hiss, Broué - Michel] Each ℓ-block of $\mathbf{G}^F$ corresponds to the conjugacy class of a unique ℓ' semisimple element s in the fixed points of the dual group, $\mathbf{G}^{*F^*}$.

$$B \mapsto (s)$$

II. When $B \mapsto 1$

- B is called a *unipotent* block
- B contains an irreducible component of $R_{\mathbf{T}}^{\mathbf{G}}(1)$ for some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$

Theorem (F.)

If B is a unipotent block of a finite group of Lie type in non-defining characteristic then $mf(B) = 1$ except for two cases in E_8 when $mf(B) \leq 2$.

III. Isolated $s \neq 1$

- A semisimple element $s \in \mathbf{G}^{*F^*}$ is *isolated* if there does not exist a proper Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$ such that $C^\circ_{\mathbf{G}^*}(s) \subset \mathbf{L}^*$
- If B corresponds to an isolated $s \in \mathbf{G}^{*F^*}$ then B is called an *isolated* block

Lemma (Lusztig)

Let $s \in \mathbf{G}^{*F^*}$ be a semisimple element. Suppose that

- *$Z(\mathbf{G})$ is connected, and*
- *All the components of $C_{\mathbf{G}^*}(s)$ are of classical type.*

Then every $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ is uniquely determined by its uniform proj.,

$$\{\langle \chi, R^{\mathbf{G}}_{\mathbf{T}}(\theta)\rangle \mid \mathbf{T} \text{ an } F\text{-stable max. torus of } \mathbf{G},\ \theta \in Irr(\mathbf{T})\}.$$

2. Blocks of finite groups of Lie type in non-defining char.

Proposition

Let $s \in \mathbf{G}^{*F^*}$ be an ℓ' semisimple element. Suppose that $C_{\mathbf{G}^*}(s)$ has components all of classical type. Let $m = o(s)$. Then for every ℓ-block B corresponding to s, $mf(B) \leq \varphi(m)$, where φ is the Euler totient function.

- [Bonnafé, 2005] Know $o(s)$ and $C_{\mathbf{G}^*}(s)$ for isolated elements $s \in \mathbf{G}^{*F^*}$

Theorem (F.)

Let B be an isolated ℓ-block of $\mathbf{G}^F$.

- *If $\mathbf{G}$ is of classical type then $mf(B) = 1$.*
- *If $\mathbf{G}$ is of exceptional type not equal to E_8, then $mf(B) \leq 2$.*

- There are two conjugacy classes of isolated elements in E_8 for which $C_{\mathbf{G}^*}(s)$ has a non-classical component
- For all other isolated blocks of E_8, $mf(B) \leq 4$

IV. Bonnafé - Rouquier, Bonnafé - Dat - Rouquier

Theorem (Bonnafé - Dat - Rouquier 2015)

Suppose that B is an ℓ-block of $\mathbf{G}^F$ corresponding to an ℓ' semisimple element s of $\mathbf{G}^{*F^*}$. Let

- $\mathbf{L}_s^*$ *be the smallest Levi subgroup of* $\mathbf{G}^*$ *containing* $C^\circ_{\mathbf{G}^*}(s)$
- $\mathbf{N}_s^* = C_{\mathbf{G}^*}(s)^{F^*}.\mathbf{L}_s^*$
- $\mathbf{L}_s$ *be the dual group of* $\mathbf{L}_s^*$
- $\mathbf{N}_s \subset \mathbf{G}$ *be such that* $\mathbf{L}_s \lhd \mathbf{N}_s$ *and* $\mathbf{N}_s/\mathbf{L}_s \cong \mathbf{N}_s^*/\mathbf{L}_s^*$ *via the canonical isomorphism from* $N_{\mathbf{G}^*}(\mathbf{L}_s^*)/\mathbf{L}_s^* \to N_{\mathbf{G}}(\mathbf{L}_s)/\mathbf{L}_s$

Then there exists an ℓ-block b of $\mathbf{N}_s^F$ which is Morita equivalent to B such that b covers an ℓ-block c of $\mathbf{L}_s^F$ which corresponds to s.

V. BDR Best Case Scenario

Proposition

Let $s \in \mathbf{G}^{*F^*}$ be an ℓ' semisimple element. Suppose that

- *$C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$,*
- *All components of $C^\circ_{\mathbf{G}^*}(s)$ are of classical type, and*
- *$C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ is cyclic.*

Then for every ℓ-block B corresponding to s, $mf(B) = 1$.

- $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ cyclic except when $\mathbf{G} = Spin^+_{2n}$ and $n \geq 4$ is even
- All components of proper Levi subgroups of $\mathbf{G}^*$ have classical type except when $\mathbf{G} = E_7$ or E_8

VI. Type A

- $C^{\circ}_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$ for any ℓ' semisimple $s \in {\mathbf{G}^*}^{F^*}$
- In particular, the Rationality Conjecture holds for Type A
- [Hiss - Kessar, 2003] Weak Donovan holds for SL_n

Theorem (F.)

Donovan's Conjecture holds for SL_n.

VII. Some Outstanding Cases

Blocks B corresponding to an ℓ' semisimple element $s \in \mathbf{G}^{*F^*}$ where

1. $C^\circ_{\mathbf{G}^*}(s)$ is a Levi subgroup with all classical components BUT $C_{\mathbf{G}^*}(s)/C^\circ_{\mathbf{G}^*}(s)$ not cyclic
    - Idea: adapt the Clifford theory in the "Best Case Scenario" to get from $\mathbf{L}_s^F$ to $\mathbf{N}_s^F$ even though $\mathbf{N}_s^F/\mathbf{L}_s^F$ is not cyclic
2. $C^\circ_{\mathbf{G}^*}(s)$ is not a Levi subgroup, s is not isolated
    - Idea: Use BDR and information about isolated blocks of Levi subgroups of $\mathbf{G}$

Thank you