Representation Theory Exercises — Sheet 10

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Due date: **Mo, 28.01.19, 12:00** — WS 18/19

In this exercise sheet G denotes a finite group and K denotes an algebraically closed field of characteristic p.

Exercise 40: Prove the following.

(a) If U and V are KG-modules and U is relatively H-projective for a subgroup H of G, then $U \otimes V$ is also relatively H-projective.
[Hint: prove the claim first for relatively free modules]

(b) Assume that a Sylow p-subgroup $P \leq G$ is a TI-subgroup of G, and set $L := N_G(P)$. Let M and N be indecomposable non-projective KG-modules. Prove that the Green correspondent of $M \otimes N$ (modulo projectives) is the tensor product of the Green correspondents of M and N.

Exercise 41: Let $V_{d+1} \subset K[X,Y]$ be the subspace of homogeneous polynomials of degree d, for $0 \leq d \leq p-1$. Let $W_i \leq V_{d+1}$ be the subspace spanned by $\{X^iY^{d-i}, X^{i-1}Y^{d-i+1}, \ldots, Y^d\}$, for $0 \leq i \leq d$. Let

$$U := \langle \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \rangle \leq G := \mathrm{SL}_2(p),$$

and let

$$B := \{ \left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{F}_p^\times, b \in \mathbb{F}_p \}.$$

Prove that the following assertions hold.

(a) W_i is a KU-submodule of V_{d+1}.

(b) W_i/W_{i-1} is the trivial KU-module.

(c) Any element of $W_i \setminus W_{i-1}$ generates W_i as a KU-module.

(d) W_i is an indecomposable KU-module with a simple head. Determine the KB-module structure of $\mathrm{hd}(W_i)$.

(e) V_{d+1} is a simple KG-module.

Exercise 42: Determine the kernels of the representations corresponding to the $K\mathrm{SL}_2(p)$-modules V_d from the previous question, for $0 \leq d \leq p$.

Exercise 43: Let F be a field, A an F-algebra, and N an algebraically closed field containing F. Show that

(a) A splits if extension of scalars induces a bijection between isomorphism classes of simple A-modules and simple A_N-modules;

(b) there is a finite field extension $F_1|F$ such that A_{F_1} splits;

(c) if A splits, then $\mathrm{Hom}_A(S,S) = F$ for every simple A-module S.