**Representation Theory Exercises — Sheet 2**

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Due date: **Mo, 12.11.18, 12:00** — WS 18/19

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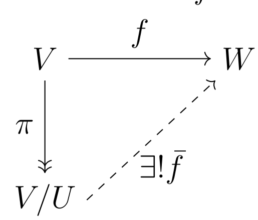
Throughout, $R$ is a commutative ring, $A$ an $R$-algebra, and all modules are assumed to be left modules.

**Exercise 5:** (a) Let $f : V \to W$ be an $A$-homomorphism of $A$-modules. Suppose that $U$ is an $A$-submodule of $V$ such that $U \leq \ker(f)$. Show that there exists a unique $A$-homomorphism $\bar{f} : V/U \to W$ such that $\bar{f}\pi = f$.




(b) Let $V$ be an $A$-module with $A$-submodules $V'$, $V''$ and $U$ such that $V' \leq V''$. Show that there exists a short exact sequence of $A$-modules and $A$-module homomorphisms

$$0 \to (V'' \cap U)/(V' \cap U) \to V''/V' \to (V'' + U)/(V' + U) \to 0.$$

**Exercise 6:** Let $K$ be a field and let $p \in K[X]$ be a non-constant, irreducible polynomial. Prove that

$$R := \left\{ \frac{f}{g} : f, g \in K[X],\, p \nmid g \right\} \subset K[X]$$

is a local ring.

**Exercise 7 (Splitting Lemma):** Let $M, N$ be two $A$-modules, and let $f : M \to N$, $g : N \to M$ be two morphisms of $A$-modules such that $gf = \mathrm{id}_M$. Then

$$N = \mathrm{Im}(f) \oplus \ker(g).$$

Moreover, $M$ is isomorphic to a direct summand of $N$.

**Exercise 8:** An $A$-module is called *Artinian* if there are no infinite strictly descending chains of submodules. It is called *Noetherian* if there are no infinite strictly ascending chains of submodules. Prove the following:

(a) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of $A$-modules, then $M$ is Artinian (Noetherian) if and only if both $M'$ and $M''$ are Artinian (Noetherian).

(b) $M$ is an Artinian and Noetherian $A$-module if and only if $M$ has a (finite) composition series.