**Representation Theory Exercises — Sheet 4**

Prof. Dr. G. Malle | Dr. N. Farrell
Due date: **Mo, 26.11.17, 12:00** | WS 18/19

---

Throughout, $A$ is an algebra over a commutative ring and all modules are assumed to be left modules.

**Exercise 12:** Let $R$ be an arbitrary ring. Prove that $\mathrm{End}_R(R) \cong R^{\mathrm{op}}$, where $R^{\mathrm{op}}$ is obtained from $R$ by replacing the multiplication $\cdot$ of $R$ with the operation $*$ defined by

$$a * b := b \cdot a.$$

**Exercise 13:** Two idempotents $e, f \in A$ are called *orthogonal* if $ef = fe = 0$. An idempotent is *primitive* if it cannot be written as the sum of two orthogonal idempotents. Show that the following are equivalent:

(i) $e$ is primitive,

(ii) $Ae$ is an indecomposable $A$-module,

(iii) $e$ is the unique idempotent of $eAe$.

**Exercise 14:** We say that the $A$-module $U$ is *injective* if every short exact sequence of $A$-modules

$$0 \to U \to V \to W \to 0$$

splits. Show that the following assertions are equivalent.

(a) $U$ is injective,

(b) for every $A$-module $V$ such that $U \leq V$, there exists an $A$-module $W \leq V$ such that $V = U \oplus W$,

(c) if $\varphi : M \to N$ is an injective morphism of $A$-modules, and $\psi : M \to U$ is a morphism of $A$-modules, then there exists a morphism of $A$-modules $\rho : N \to U$ such that $\rho\varphi = \psi$.

**Exercise 15:** Let $K$ be a field and $A$ the (associative) $K$-algebra with basis $\{1, i, j, k\}$ and multiplication $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

(a) Show that $A$ is a skew field if and only if $X^2 + Y^2 + Z^2 + 1 = 0$ has no solutions over $K$.

(b) Construct the left regular representation $V := A^\circ$ for $A$ acting on itself by determining matrices for $1, i, j, k$.

(c) Determine $\mathrm{End}_A(V)$ and $Z(A)$. Under which conditions is $V$ irreducible?