Representation Theory Exercises — Sheet 5

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Due date: **Mo, 03.12.18, 12:00** — WS 18/19

Throughout, A is an algebra over a commutative ring, K denotes an algebraically closed field, and all modules are assumed to be left modules.

Exercise 16: Let V be an A-module, $e \in A$ an idempotent. Prove that

$$\mathrm{Hom}_A(Ae, V) \cong eV$$

as $\mathrm{End}_A(V)$-modules.

Exercise 17: Let A be a finite-dimensional K-algebra. Let P be a finite-dimensional, projective, indecomposable A-module and let $S := P/\mathrm{rad}(P)$ be the corresponding simple A-module. Prove that the multiplicity of S as a composition factor in any A-module V is given by $\dim_K(\mathrm{Hom}_A(P, V))$.

Exercise 18: Let G be a finite group, $H \leq G$ a subgroup of index $|G : H|$ prime to the characteristic of K. Let V be a KG-module which is completely reducible as a KH-module. Prove that V is completely reducible as a KG-module. (This generalizes Maschke's theorem.)

Exercise 19: Let $\phi : \mathfrak{S}_n \to \mathrm{GL}_n(K)$ be the natural permutation representation of the symmetric group $\mathfrak{S}_n$ on $V = K^n$.

(a) Prove that $V_1 := \{(v_i)_i \mid \sum_i v_i = 0\}$ and $V_2 := \{(v)_i \mid v \in K\}$ are $\mathfrak{S}_n$-invariant subspaces of V.

(b) Determine under which conditions we have $V = V_1 \oplus V_2$.

(c) Give the character values of the corresponding representation of $\mathfrak{S}_n$ on V_1.