Representation Theory Exercises — Sheet 6

Prof. Dr. G. Malle — Dr. N. Farrell

Due date: **Mo, 10.12.18, 12:00** — WS 18/19

Throughout, G denotes a finite group, and K a field of arbitrary characteristic. All KG-modules are assumed to be finitely generated, left KG-modules.

Exercise 20: Compute the complex character table of the alternating group $\mathfrak{A}_4$ as follows:

(i) Determine the conjugacy classes of $\mathfrak{A}_4$ (there are 4 of them) and the corresponding centraliser orders.

(ii) Determine the degrees of the 4 irreducible characters of $\mathfrak{A}_4$.

(iii) Determine the linear characters of $\mathfrak{A}_4$.

(iv) Determine the non-linear character of $\mathfrak{A}_4$ using the 2nd Orthogonality Relations.

Exercise 21: Let G be a finite group and K a field whose characteristic divides $|G|$. *Without using* Corollary 4.22, prove that KG is not semisimple.
[Hint: check first that $(\sum_{g\in G} g)^2 = 0$.]

Exercise 22: Let V, W be KG-modules. Prove that:

(a) $V \cong (V^*)^*$ as KG-modules (in a natural way).

(b) $V^* \oplus W^* \cong (V \oplus W)^*$ as KG-modules (in a natural way).

(c) If $\rho : W \to V$ is an injective (resp. surjective) KG-homomorphism, then $\rho^* : V^* \to W^*$ is surjective (resp. injective). Thus, if $W \subseteq V$ is a KG-submodule, there exists a KG-submodule $M \subseteq V^*$ such that $M \cong (V/W)^*$ and $V^*/M \cong W^*$.

(d) The set $\mathrm{PHom}_{KG}(V, W)$ of projective KG-homomorphisms from V to W is a K-subspace of $\mathrm{Hom}_{KG}(V, W)$.

Exercise 23: Let V be a KG-module. Prove that:

(a) Every finitely generated KG-module has an injective hull.
[Hint: Use a projective cover of V^*]

(b) If P is an injective hull of V, then there exists an injective KG-homomorphism $V \longrightarrow P$, which induces an isomorphism $\operatorname{soc}(P) \cong \operatorname{soc}(V)$ by restriction.

(c) If P is an injective hull of V, $\varphi : V \longrightarrow P$ is an injective KG-homomorphism, and $\psi : V \longrightarrow Q$ is an injective KG-homomorphism into an injective KG-module Q, then the following holds:

   (i) there exists a KG-homomorphism $\rho : P \longrightarrow Q$ such that $\psi = \rho \circ \varphi$;

   (ii) there is a direct sum decomposition $Q = R \oplus \operatorname{coker}(\rho)$, where $R \cong P$.

(d) An injective hull of V is uniquely determined up to KG-isomorphism.