Representation Theory Exercises — Sheet 7

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Due date: **Mo, 17.12.17, 12:00** | WS 18/19

Let K be a field and G be a finite group. In this exercise sheet, we write $\otimes$ for $\otimes_K$.

Exercise 25: Check that the distributivity of multiplication over addition holds in $K_0(KG)$, and determine whether or not $K_0(KG)$ has a neutral element for the multiplication.

Exercise 26 (Universal property of the tensor product): Let V, W, U be finite dimensional K-vector spaces. Prove that:

(a) For every K-bilinear map $\varphi : V \times W \to U$, there exists a uniquely determined K-homomorphism $\psi : V \otimes W \to U$ such that $\varphi = \psi \circ t$, where

$$t : V \times W \to V \otimes W, \qquad (v, w) \mapsto v \otimes w,$$

is the natural map.

(b) The tensor product $V \otimes W$ is uniquely determined (up to K-isomorphism) by the property in (a).

Exercise 27: Let V, W be KG-modules. Prove the following assertions:

(a) If $V_1 \leq V$ is a KG-submodule, then $V_1 \otimes W \leq V \otimes W$ in a natural way, and $(V \otimes W)/(V_1 \otimes W) \cong V/V_1 \otimes W$.

(b) We have $V^* \otimes W^* \cong (V \otimes W)^*$ as KG-modules.

Exercise 28: Let U, V be KG-modules.

(a) Prove that $\Phi_{U,V} : U^* \otimes V \to \mathrm{Hom}_K(U, V)$ defined by $\Phi_{U,V}(\varphi \otimes v)(u) := \varphi(u)v$ for $\varphi \in U^*$, $v \in V$, $u \in U$, is a KG-isomorphism.

(b) Let $\mathrm{Tr}_V : V^* \otimes V \to K, (f, v) \mapsto f(v)$ be the *trace map*. Prove that Tr_V is a KG-homomorphism and that $\mathrm{Tr}_V \circ (\Phi_{V,V})^{-1}$ coincides with the ordinary trace of matrices.

(c) Prove that V is a direct summand of $V \otimes V^* \otimes V$. Moreover, if $\mathrm{char}(K) = p > 0$ and $p \,|\, \dim_K V$, then $V \oplus V$ is a direct summand of $V \otimes V^* \otimes V$.