Representation Theory Exercises — Sheet 9

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Due date: **Mo, 21.01.19, 12:00** | WS 18/19

In this exercise sheet, G denotes a finite group.

Exercise 35: Let K be a field. Let V be a simple KG-module and $N \trianglelefteq G$. Let W be a simple KN-submodule of V. Prove that every KN-module conjugate to W occurs with the same multiplicity as W as a KN-submodule of V.

Exercise 36 (Relative projectivity with respect to a KG-module):
Let K be a field and let V be a fixed KG-module. A KG-module M is called V*-projective* if there exists a KG-module N such that M is isomorphic to a direct summand of $V \otimes N$ (as usual, $\otimes$ denotes the tensor product over K). Denote by $\mathrm{Proj}(V)$ the collection of all V-projective KG-modules. Let U be another KG-module.

(a) Prove that M is projective relative to the subgroup $H \leq G$ if and only if M is K^G-projective, where K is the trivial KH-module.

(b) Prove that:

  (i) $\mathrm{Proj}(U \oplus V) = \mathrm{Proj}(U) \oplus \mathrm{Proj}(V)$;

  (ii) $\mathrm{Proj}(U \otimes V) = \mathrm{Proj}(U) \cap \mathrm{Proj}(V)$; and

  (iii) $\mathrm{Proj}(V) = \mathrm{Proj}(V^*) = \mathrm{Proj}(V \oplus V)$.

(c) Prove that $M \in \mathrm{Proj}(V)$ if and only if M is a direct summand of $V^* \otimes V \otimes M$.
[Hint: remember the trace map introduced in Exercise 28]

(d) Prove that if $K \in \mathrm{Proj}(V)$, then every KG-module M is V-projective.

Exercise 37: Suppose that K is an algebraically closed field of characteristic $p > 0$. Let $H \trianglelefteq G$, and let V be an indecomposable KH-module with *inertia subgroup* T, that is,

$$T = \{g \in G \mid gV = V\}.$$

Prove that the following are equivalent:

(a) V^G is an indecomposable KG-module.

(b) V^T is an indecomposable KT-module.

(c) T/H is a p-group.

Exercise 38: Let K be a field of characteristic $p > 0$, and let V be an indecomposable KG-module of dimension not divisible by p. Prove that the vertices of V are the Sylow p-subgroups of G.

Exercise 39: Let K be a field, let $H \leq G$, and let V be an indecomposable KG-module such that $V \big| (K_H)^G$. Prove that V has a trivial source.