Modular Representation Theory of Finite Groups

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Chapter 9. Character Theory and Decomposition Matrices

The goal of this chapter is to define a character theory for modular representations of finite groups and to use character theory to learn more about the decomposition matrices of finite groups.

Notation: Throughout, G denotes a finite group and $(K, \mathcal{O}, k)$ is a splitting p-modular system for G.

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[Nav98] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series, vol. 250, Cambridge University Press, Cambridge, 1998.

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35 Ordinary Characters

In this section we will briefly review some important definitions and results from ordinary character theory. We work over K, a splitting field for G of characteristic 0. In particular, KG is semisimple. The character theory of G over K is the same as the character theory of G over $\mathbb{C}$, which you have probably seen in an earlier course.

Definition 35.1

Let $\rho : G \to GL(V)$ be a K-representation of G for some $V \cong K^n$, $n \geqslant 1$. Then

$$\begin{aligned} \chi : G &\to K \\ g &\mapsto \operatorname{tr}(\rho(g)), \end{aligned}$$

is the **character of** ρ, or the **character afforded by** ρ. If ρ has degree one then χ is called a **linear** character. If ρ is an irreducible representation then χ is called an **irreducible** character. We denote the set of irreducible characters of G by $\operatorname{Irr}(G)$.

Remark 35.2

- Let $X : G \to \mathrm{GL}_n(K)$ be a matrix representation of G of degree $n \geqslant 1$. Then

$$\begin{aligned}\chi : G &\to K\\ g &\mapsto \mathrm{tr}(X(g))\end{aligned}$$

is the **character of** X or the **character afforded by** X.

- **Exercise 35.3**

  Show that two similar matrix representations afford the same character.

Exercise 35.4

Lemma 35.5

Let $X : G \to \mathrm{GL}_n(\mathbb{C})$ be a complex representation of G of degree $n \geqslant 1$ and let χ be the character afforded by X.

(a) $\chi(1) = n$.

(b) $\chi(g)$ is a sum of $o(g)$-th roots of unity for all $g \in G$.

(c) $|\chi(g)| \leqslant \chi(1)$ for all $g \in G$.

(d) $\overline{\chi}$ is also a character of G, defined by $\overline{\chi}(g) = \chi(g^{-1})$ for all $g \in G$.

(e) $\chi(g) = \chi(h^{-1}gh)$ for all $g, h \in G$, i.e. characters are class functions.

Notation 35.6

Let $C_1, \ldots, C_d$ be the conjugacy classes of G and denote the class sums by

$$\hat{C}_i := \sum_{g \in C_i} g$$

for each $1 \leqslant i \leqslant d$. Let $\mathrm{Cl}(G)$ denote the set of complex valued class functions of G.

Theorem 35.7

The class sums $\hat{C}_i, \ldots, \hat{C}_d$ are a basis for $Z(KG)$.

Proof: Let $h \in G$. Then for any $1 \leqslant i \leqslant d$,

$$h\hat{C}_i = \sum_{g \in C_i} hg = \sum_{g \in C_i} hgh^{-1}h = \hat{C}_i h$$

since as g runs over C_i, so does hgh^{-1}. Hence $\hat{C}_i \in Z(KG)$ for any $1 \leqslant i \leqslant d$. Since $\{g \in G\}$ is a basis for KG, the set of class sums $\{\hat{C}_i\}_{1 \leqslant i \leqslant d}$ is linearly independent since they are sums of disjoint sets of elements of G.

Let $h \in G$ and let $z \in Z(KG)$ such that $z = \sum_{g \in G} a_g g$. Then

$$\sum_{g \in G} a_g g = z = h^{-1} z h = \sum_{g \in G} a_g h^{-1} g h = \sum_{g \in G} a_g g^h.$$

Equating coefficients in the sums shows that a_g is constant on conjugacy classes so $z = \sum_{i=1}^{d} a_{g_i} \hat{C}_i$, where $g_i \in C_i$. In particular, $Z(KG) \subseteq \langle \hat{C}_i \rangle$ and hence $\hat{C}_1, \ldots, \hat{C}_r$ is a basis for $Z(KG)$. ■

Remark 35.8

[Edited 19.01.20] Two irreducible representations with the same character are similar. Thus by the arguments in Chapter 3 we have the following bijections

$$\mathrm{Irr}(G) \leftrightarrow \left\{ \begin{array}{c} \text{Irreducible } K\text{-reps of } G \\ \text{up to equivalence} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Simple } KG\text{-modules} \\ \text{up to isomorphism.} \end{array} \right\}$$

and since KG is semisimple, these sets are of size $\dim_K(Z(KG))$ by Corollary 13.4.

Corollary 35.9

The number of conjugacy classes of G is equal to $|\mathrm{Irr}(G)|$.

Proof: Immediate from Remark 35.8 and Theorem 35.7. ■

Definition 35.10

The **regular character** is the character χ_{reg} afforded by the regular representation ρ_{reg} of G (see Section 14, Example 8).

Lemma 35.11

For any $g \in G$,

$$\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$$

Proof: Let $g \in G$. Then $\rho_{\mathrm{reg}}(g) = (a_{hk})_{h,k \in G}$ where

$$a_{hk} = \begin{cases} 1 & \text{if } hg = k \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\chi_{\mathrm{reg}}(g) = \mathrm{tr}(\rho_{\mathrm{reg}}(g)) = \#\{h \in G \mid hg = h\} = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise} \end{cases}$ ■

Proposition 35.12

We have $\chi_{\mathrm{reg}} = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)\chi$.

Proof: By Theorem 13.2, since K is a splitting field for G and KG is semisimple, every irreducible representation X of G appears in the regular representation ρ_{reg} exactly $\dim_K(X)$ times. The result follows. ■

Exercise 35.13

Corollary 35.14 (*Degree forumla*)

Let $\mathrm{Irr}(G) = \{\chi_1, \dots, \chi_d\}$. Then $|G| = \sum_{i=1}^{d} \chi_i(1)^2$.

Notation 35.15

Let $\chi \in \mathrm{Irr}(G)$ and let X be an irreducible representation of G affording χ. Let S be the simple module corresponding to X as in Proposition 15.3. Then fix $e_\chi := e_S$, where the latter is the central primitive idempotent associated to S as in Scholium 12.6.

Remark 35.16

We can linearly extend a matrix representation of G to a representation of the group algebra KG,

$$X : KG \to M_n(K).$$

The character of X is defined by $\chi : KG \to K$, $\chi(g) = \mathrm{tr}(X(g))$ for all $g \in G$ and its restriction to G is just a character of G. We can therefore consider characters acting on elements of KG and not just on elements of G.

Proposition 35.17

For any $\chi \in \mathrm{Irr}(G)$, we have

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(1)\chi(g^{-1})g.$$

Proof: Write $e_\chi = \sum_{g\in G} a_g g$. By Lemma 35.11, we have for any $g \in G$,

$$\chi_{\mathrm{reg}}(e_\chi g^{-1}) = \chi_{\mathrm{reg}}\left(\sum_{h\in G} a_h h g^{-1}\right) = \sum_{h \in G} a_h \chi_{\mathrm{reg}}(hg^{-1}) = a_g|G|.$$

On the other hand, Proposition 35.12 shows that

$$\chi_{\mathrm{reg}}(e_\chi g^{-1}) = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)\psi(e_\chi g^{-1}).$$

Now $e_\chi g^{-1} \in e_\chi KG$, so by the orthogonality of the idempotents, $e_\chi g^{-1}$ is in the kernel of ψ for all $\psi \in \mathrm{Irr}(G)$ such that $\psi \neq \chi$. Therefore $a_g|G| = \chi(1)\chi(e_\chi g^{-1})$. But the idempotent e_χ is the identity in $e_\chi KG$, so $\chi(e_\chi g^{-1}) = \chi(g^{-1})$ for all $g \in G$, $\chi \in \mathrm{Irr}(G)$. Hence

$$e_\chi = \sum_{g\in G} a_g g = \sum_{g \in G} \frac{\chi(1)\chi(e_\chi g^{-1})}{|G|} g = \frac{1}{|G|}\sum_{g\in G} \chi(1)\chi(g^{-1})g,$$

as claimed. ■

Theorem 35.18 (*First orthogonality relations*)

For all $h \in G$ and all $\chi, \psi \in \mathrm{Irr}(G)$,

$$\frac{1}{|G|}\sum_{g\in G} \chi(gh)\psi(g^{-1}) = \delta_{\chi\psi}\frac{\chi(h)}{\chi(1)}$$

In particular, for $h = 1$ we have

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}) = \begin{cases} 1 & \text{if } \chi = \psi \\ 0 & \text{otherwise} \end{cases}$$

Proof: Let e_χ and e_ψ be the central primitive idempotents associated to the irreducible characters χ and ψ as in Notation 35.15. Since they are orthogonal idempotents, $e_\chi e_\psi = \delta_{\chi\psi}e_\chi$. Hence from the formula given in Proposition 35.17, we have

$$|G|e_\chi e_\psi = \frac{1}{|G|}\sum_{h\in G}\sum_{g\in G}\chi(1)\psi(1)\chi(h^{-1})\psi(g^{-1})hg = \delta_{\chi\psi}|G|e_\chi = \delta_{\chi\psi}\sum_{k\in G}\chi(1)\chi(k^{-1})k.$$

Comparing coefficients for $k \in G$ and dividing by $\chi(1)$ shows that

$$\frac{1}{|G|}\sum_{g,h\in G, hg=k}\psi(1)\chi(h^{-1})\psi(g^{-1}) = \delta_{\chi\psi}\chi(k^{-1}).$$

Hence with $h = kg^{-1}$,

$$\frac{1}{|G|}\sum_{g\in G}\psi(1)\chi(gk^{-1})\psi(g^{-1}) = \delta_{\chi\psi}\chi(k^{-1}).$$

Now writing h instead of k throughout we get the desired result. ■

Proposition 35.19

The map

$$\begin{aligned} \langle\ ,\ \rangle : \mathrm{Cl}(G)\times\mathrm{Cl}(G) &\to \mathbb{C} \\ (\chi,\psi) &\mapsto \langle\chi,\psi\rangle := \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}) \end{aligned}$$

is a symmetric $\mathbb{C}$-bilinear form. The irreducible characters $\mathrm{Irr}(G)$ form an orthonormal basis for $\mathrm{Cl}(G)$ with respect to $\langle\ ,\ \rangle$. Further, $\langle\ ,\ \rangle$ is positive definite.

Proof:
- Bilinearity: Exercise
- Symmetry: Exercise
- $\mathbb{C}$-linear independence of $\mathrm{Irr}(G)$: Exercise
- $\mathrm{Irr}(G)$ is an orthonormal basis for $\mathrm{Cl}(G)$: Exercise
- Positive definiteness: Exercise ■

Corollary 35.20

(a) Let f be a class function of G. Then $f = \sum_{\chi\in\mathrm{Irr}(G)}\langle f,\chi\rangle\chi$.

(b) A character χ of G is irreducible if and only if $\langle\chi,\chi\rangle = 1$.

Proof: (a) For $\psi \in \mathrm{Irr}(G)$ we have

$$\langle\sum_{\chi\in\mathrm{Irr}(G)}\langle f,\chi\rangle\chi,\psi\rangle = \sum_{\chi\in\mathrm{Irr}(G)}\langle f,\chi\rangle\langle\chi,\psi\rangle = \langle f,\psi\rangle$$

by the first orthogonality relations. Hence

$$\langle f - \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle, \psi \rangle = 0$$

for all $\psi \in \mathrm{Irr}(G)$. Therefore $f - \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle = 0$ by Proposition 35.19, so $f = \sum_{\chi \in \mathrm{Irr}(G)} \langle f, \chi \rangle \chi$.

(b) Since $\mathrm{Irr}(G)$ is a basis for the class functions of G by Proposition 35.19, and characters are class functions by Lemma 35.5 (e), ψ is a character of G if and only if

$$\psi = \sum_{\chi \in \mathrm{Irr}(G)} n_\chi \chi$$

for some $n_\chi \geqslant 0$. Thus,

$$\langle \psi, \psi \rangle = \sum_{\chi, \mu} n_\chi n_\mu \langle \chi, \mu \rangle = \sum_{\chi, \mu} n_\chi n_\mu \delta_{\chi\mu} = \sum_{\chi, \mu} n_\chi^2.$$

Therefore $\langle \psi, \psi \rangle = 1$ if and only if there exists a unique $\chi \in \mathrm{Irr}(G)$ with $n_\chi = 1$ and $n_\varphi = 0$ for all $\varphi \in \mathrm{Irr}(G)$ such that $\varphi \neq \chi$. In other words, $\langle \psi, \psi \rangle = 1$ if and only if $\psi = \chi \in \mathrm{Irr}(G)$. ■

Theorem 35.21 (*Second orthogonality relations*)

For all $g, h \in G$, we have

$$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\chi(h^{-1}) = \begin{cases} |C_G(g)| & \text{if } g \text{ is conjugate to } h \\ 0 & \text{otherwise} \end{cases}$$

Proof: The first orthogonality relations (Theorem 35.18) shows that since characters are class functions, for any $\chi, \psi \in \mathrm{Irr}(G)$ we have

$$\delta_{\chi\psi}|G| = \sum_{g \in G} \chi(g)\psi(g^{-1}) = \sum_{i=1}^{d} \chi(g_i)|C_i|\psi(g_i^{-1}),$$

where $g_1, \ldots, g_d$ is a set of representatives of the conjugacy classes $C_1, \ldots, C_d$ of G. Define the following $d \times d$ matrices:

$$\begin{aligned} I_d &:= \text{ the identity matrix} \\ X &:= (\chi(g_i))_{\chi \in \mathrm{Irr}(G), 1 \leqslant i \leqslant d} \\ \tilde{X} &:= (\chi(g_i^{-1}))_{\chi \in \mathrm{Irr}(G), 1 \leqslant i \leqslant d} \\ D &:= \text{ diag } (|C_1|, \ldots, |C_d|) \end{aligned}$$

Then the equation above can be expressed as

$$|G|I_d = XD\tilde{X}^t,$$

so $\frac{1}{|G|}X$ is a left inverse of $D\overline{X}^t$, and therefore also a right inverse so we have

$$|G|I_d = D\tilde{X}^t X,$$

which gives, for each $1 \leqslant i, j \leqslant d$,

$$|G|\delta_{ij} = \sum_{\chi \in \mathrm{Irr}(G)} |C_i|\chi(g_i^{-1})\chi(g_j).$$

Hence, since $\frac{|G|}{|C_i|} = |C_G(g_i)|$, we have

$$\delta_{ij}|C_G(g_i)| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(g_i^{-1})\chi(g_j)$$

for all $g_i \in C_i$. ■

Remark 35.22

Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G for some $V \cong K^n$, $n \geqslant 1$, and define a KG-module structure on V as in Proposition 15.3. Since KG is semisimple, $V = \oplus_{i=1}^r S_i$ for simple KG-modules S_i. Therefore any matrix representation X associated to ρ is similar to a diagonal representation

$$X = \begin{pmatrix} X_1 & 0 & \dots & 0 \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & X_r \end{pmatrix},$$

where X_i are irreducible representations of G called the **irreducible constituents** of X. Let $\chi_i \in \mathrm{Irr}(G)$ denote the character of X_i for each $1 \leqslant i \leqslant r$. Then $\sum_{i=1}^r \chi_i$ is the character of X.

Proposition 35.23

Representations with the same character are similar.

Proof: Let X and X' be representations of G with characters $\sum_{i=1}^r \chi_i$ and $\sum_{j=1}^s \chi_j'$ respectively, where $\chi_i, \chi_j' \in \mathrm{Irr}(G)$ for all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$. Then since the irreducible characters of G are linearly independent, if $\sum_{i=1}^r \chi_i = \sum_{j=1}^s \chi_j'$ then $r = s$ and without loss of generality, $\chi_i = \chi_i'$ for all $1 \leqslant i \leqslant r$. Thus each of the irreducible constituents X_i of X is similar to the corresponding irreducible constituent X_i' of X', so X is similar to X'. ■

36 Brauer Characters

Notation 36.1

We will now fix a particular splitting p-modular system for G.

- Let $X_1, \dots, X_r$ be a complete system of representatives for the isomorphism classes of irreducible representations of G over a splitting field of finite degree over $\mathbb{Q}_p$.
- Let $Y_1, \dots, Y_s$ be a complete system of representatives for the isomorphism classes of irreducible representations of G over a splitting field of finite degree over $\mathbb{F}_p$.
- Let $k_1 \mid \mathbb{F}_p$ be generated by a $|G|_{p'}$- root of unity and the entries in the matrices $Y_i(g)$ for all $g \in G$, $1 \leqslant i \leqslant s$. Then k_1 is a finite extension of $\mathbb{F}_p$, so $k_1 = \mathbb{F}_q$ for some $q = p^f$, $f \geqslant 1$.
- Let $K \mid \mathbb{Q}_p$ be generated by a $(q$-$1)$th root of unity and the entries in the matrices $X_i(g)$ for all $g \in G$, $1 \leqslant i \leqslant r$.
- Let $\mathcal{O}$ be the integral closure of $\mathbb{Z}_p$ in K.

- Define $k := \mathcal{O}/J(\mathcal{O})$.

It is possible to show that the ring $\mathcal{O}$ is a complete discrete valuation ring. Thus $J(\mathcal{O})$ is its unique maximal ideal. The residue class k contains $k_1 = \mathbb{F}_q$, and both K and k are splitting fields for G. Thus $(K, \mathcal{O}, k)$ is a splitting p-modular system for G.

Lemma 36.2

Let X be a k-representation of G. Then the eigenvalues of $X(g)$ are contained in k for all $g \in G$.

Proof: Let $g \in G$ and write $o(g) = p^n m$ for $n \geqslant 0$ and $m \geqslant 1$ such that $(p, m) = 1$. Then $m \mid |G|_{p'}$.

Let ξ be an eigenvalue of $X(g)$. Then $\xi^{o(g)} = 1$, so

$$0 = \xi^{p^n m} - 1 = (\xi^m - 1)p^n,$$

because k has characteristic p. Therefore $\xi^m = 1$. By construction, k contains k_1 which contains the $|G|_{p'}$-th roots of unity. The result follows since $m \mid |G|_{p'}$. ■

Notation 36.3

Let $\star : \mathcal{O} \to k$ be the natural quotient map and let U denote the set of p'-roots of unity in $K^\times$.

$$U := \{\alpha \in K^\times \mid \alpha^m = 1 \text{ for an } m \in \mathbb{N} \text{ such that } (m, p) = 1\} \subseteq \mathcal{O}.$$

Proposition 36.4

The restriction of $\star$ to U defines an injective homomorphism of multiplicative groups $\star : U \to k^\times$ which is surjective on the $|G|_{p'}$th roots of unity.

Proof: First of all we will show that $J(\mathcal{O}) \cap \mathbb{Z} = p\mathbb{Z}$. It is clear that $p\mathbb{Z} \subseteq J(\mathcal{O}) \cap \mathbb{Z}$. Suppose that $m \in J(\mathcal{O}) \cap \mathbb{Z}$ is not divisible by p. Then there exist integers a and b such that $ap + bm = 1$. Therefore $1 \in J(\mathcal{O})$, which is a contradiction, so every element of $J(\mathcal{O}) \cap \mathbb{Z}$ is divisible by p and hence $J(\mathcal{O}) \cap \mathbb{Z} = p\mathbb{Z}$.

Let $1 \neq \zeta \in U$ be a primitive mth root of unity. Then

$$1 + x + x^2 + \cdots + x^{m-1} = \frac{x^m - 1}{x - 1} = \prod_{i=1}^{m-1} (x - \zeta^i)$$

Setting $x = 1$ we see that m is divisible by $1 - \zeta$. Suppose that $\zeta^\star = 1$. Then $m^\star = 0$ so $m \in J(\mathcal{O})$. But m is p' so this contradicts $J(\mathcal{O}) \cap \mathbb{Z} = p\mathbb{Z}$. Hence the only $\zeta \in U$ such that $\zeta^\star = 1$ is $\zeta = 1$, so the $\star$ map is injective on U.

Now since K contains a $(q-1)$th root of unity and $|G|_{p'}$ divides $q - 1$, it is clear that the map $\star$ is surjective onto the $|G|_{p'}$th roots of unity. ■

Definition 36.5

Denote the set of **p-regular elements** of G by

$$G^\circ := \{g \in G \mid p \nmid o(g)\}$$

Let $X : G \to \mathrm{GL}_n(k)$ be a matrix representation of G. By the setup of Notation 36.1, for any $g \in G^\circ$, the eigenvalues $\beta_1, \ldots, \beta_n$ of $X(g)$ lie in $k^\times$. Thus Proposition 36.4 shows that there exist uniquely determined roots of unity $\xi_1, \ldots, \xi_n \in U$ such that $\xi_i^\star = \beta_i$ for $1 \leqslant i \leqslant n$. The map

$$\varphi : G^\circ \to \mathcal{O}$$
$$g \mapsto \xi_1 + \cdots + \xi_n$$

is called the **Brauer Character** of the representation X of G. The **degree** of φ is n. We note the following.

- $\varphi(g) \in \mathcal{O} \subseteq \overline{\mathbb{Q}}_p$ even though $X(g) \in \mathrm{GL}_n(k)$.
- Often the values of Brauer characters are considered as complex numbers (sums of complex roots of unity). In that case then $\varphi(g)$ depends on the choice of embeddding of U into $\mathbb{C}$. For a fixed embedding, $\varphi(g)$ is uniquely determined up to similarity of X.

Definition 36.6

The Brauer character φ is **irreducible** if X is irreducible. We let $\mathrm{IBr}(G)$ denote the set of all irreducible Brauer characters of G. The Brauer character of the trivial representation $G \to \mathrm{GL}_1(k)$, $g \mapsto 1$ is denoted by 1_{G°. We say that a Brauer character λ is **linear** if $\lambda(1) = 1$.

Notation 36.7

Let $\mathrm{Cl}(G^\circ)$ denote the set of $\mathbb{C}$-valued class functions on G°.

Lemma 36.8

Let $X : G \to \mathrm{GL}_n(k)$ be a representation of G for some $n \geqslant 1$. Then for all $g \in G^\circ$, $X(g)$ is similar to a diagonal matrix $\mathrm{diag}(\xi_1^\star, \ldots, \xi_n^\star)$ for some $\xi_1, \ldots, \xi_n \in U$.

Proof: Let $g \in G^\circ$. Consider the restriction of X to the cyclic group $\langle g \rangle$. Since $\langle g \rangle$ is abelian, it follows from Corollary 17.2 that all irreducible representations of $\langle g \rangle$ have degree 1. Since $(o(g), p) = 1$, the characteristic of k does not divide $|\langle g \rangle|$ and hence by Maschke's Theorem (Theorem 16.1), $k\langle g \rangle$ is semisimple. Therefore $X(g)$ is similar to a diagonal matrix $\mathrm{diag}(\beta_1, \ldots, \beta_n)$ for some $\beta_1, \ldots, \beta_n \in k^\times$. This yields the result if we let $\xi_i \in U$ be the unique root of unity such that $\xi_i^\star = \beta_i$ for each $1 \leqslant i \leqslant n$. ■

Proposition 36.9

Let φ be a Brauer character of G.

(a) $\varphi \in \mathrm{Cl}(G^\circ)$.

(b) For any $g \in G^\circ$, $\varphi(g^{-1}) = \overline{\varphi(g)}$.

(c) The function $\overline{\varphi} : G^\circ \to \mathbb{C}$, $g \mapsto \varphi(g^{-1})$ is a Brauer character

(d) For $H \leqslant G$, $\varphi_H := \varphi\,|_{H^\circ}$ is a Brauer character of H.

Proof: Let X be a matrix representation affording φ.

(a) For any $g, h \in G^\circ$, $X(g^h) = X(h^{-1})X(g)X(h)$ so the $X(g)$ and $X(g^h)$ are similar and therefore have the same eigenvalues. Thus $\varphi(g) = \varphi(g^h)$ for any $g, h \in G^\circ$.

(b) It follows from Lemma 36.8 that for any $g \in G^\circ$, the matrix $X(g)$ is similar to a diagonal matrix $\mathrm{diag}(\xi_1^\star, \ldots, \xi_n^\star)$ for some $\xi_1, \ldots, \xi_n \in U$. Hence $X(g^{-1}) = X(g)^{-1}$ is similar to

$\mathrm{diag}((\xi_1^{-1})^\star, \ldots, (\xi_n^{-1})^\star)$. Now each ξ_i is a root of unity so $\xi_1^{-1} = \overline{\xi}_i$ for $1 \leqslant i \leqslant n$, and hence $\varphi(g^{-1}) = \overline{\varphi(g)}$ for all $g \in G^\circ$.

(c) By (b), the map $Y : G \to \mathrm{GL}_n(k)$, $g \mapsto Y(g) := X(g^{-1})^t$ is a representation with character $\overline{\varphi}$.

(d) The restriction X_H is a representation of H with character φ_H. ■

Remark 36.10

Suppose that $\rho : G \to \mathrm{GL}(V)$ is a representation of G for some $V \cong k^n$, $n \geqslant 1$, and let X be a matrix representation associated to ρ as in Remark 35.22. Since kG is not semisimple in general, we cannot conclude that X is similar to a diagonal representation. We can however, show the following.

Let $0 = V_0 < \cdots < V_r = V$ be a composition series for V for some $r \in \mathbb{N}$. Choose a basis for V_1. Extend this to a basis of V_2, and so on, until you get a basis of V. For this choice of basis, the matrix representation associated to ρ is an upper triangular block matrix of the form

$$\begin{pmatrix} X_1 & * & \dots & * \\ 0 & X_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & X_r \end{pmatrix},$$

where X_i is an irreducible representation corresponding to the simple kG-module V_i/V_{i-1} for $1 \leqslant i \leqslant r$. Hence by abuse of language, we say that X *is similar to a representation in upper block diagonal form*. It follows from Jordan-Hölder that the simple modules V_i/V_{i-1} are determined up to isomorphism by V, and hence the irreducible representations X_i are uniquely determined up to similarity by X. As in the semisimple case, the irreducible representations X_i are called the **irreducible constituents** of X.

Theorem 36.11

A class function $\varphi \in \mathrm{Cl}(G^\circ)$ is a Brauer character if and only if it is a non-negative integer linear combination of elements of $\mathrm{IBr}(G)$.

Proof: By Remark 36.10, if a class function φ is a Brauer character afforded by representation X then X is similar to a representation in upper block diagonal form and φ is just the sum of the irreducible Brauer characters afforded by the irreducible constituents of X. ■

Notation 36.12

For $g \in G$, let $g = g_p g_{p'}$ be the splitting of g into its p-part and its p'-part. Then if $o(g) = p^n m$ with $(p, m) = 1$ and $1 = ap^n + bm$, then $g_p = g^{bm}$ and $g_{p'} = g^{ap^n}$.

Proposition 36.13

Let X be a representation of kG and let ψ be the trace function on X, $\psi : kG \to k$, $\psi(g) = \mathrm{tr}\, X(g))$ for all $g \in G$. Let φ be the Brauer character afforded by X, and define $\varphi^\star : G \to k^\times$ by $\varphi^\star(g) = \varphi(g_{p'})^\star$ for all $g \in G$. Then,

(a) $\psi(g) = \psi(g_{p'})$ for all $g \in G$,

(b) $\psi(g) = \varphi(g_{p'})^\star$ for all $g \in G$, and

(c) $\{\varphi^\star \mid \varphi \in \mathrm{IBr}(G)\}$ is the set of trace functions of the irreducible kG-representations.

Proof:

(a) Since X is similar to a representation in upper block diagonal form, we can assume that X is irreducible. There is also no loss of generality if we assume that $G = \langle g \rangle$. Then all the irreducible representations of G are one dimensional. Therefore $\psi = X : G \to k^\times$ is a group homomorphism so $\psi(g) = \psi(g_p)\psi(g'_p)$. But g_p has p-power order. Therefore $\psi(g_p) \in k^\times$ also has p-power order. But in a field of characteristic p, the only element with order a power of p is 1. Therefore $\psi(g) = \psi(g_{p'})$, for all $g \in G$.

(b) This holds by definition of φ and part (a).

(c) This holds because $\varphi^\star(g) = \varphi(g_{p'})^\star = trX(g) = \psi(g)$ for all $g \in G$. ■

Theorem 36.14

The set of irreducible Brauer characters of G, $\mathrm{IBr}(G)$, is linearly independent over $\mathbb{C}$ and hence

$$|\mathrm{IBr}(G)| \leqslant \dim_{\mathbb{C}} \mathrm{Cl}(G^\circ) = \text{ The number of conjugacy classes of } p'\text{-elements in } G$$

Proof: Omitted. ■

37 Decomposition Matrices of Finite Groups

In this section we continue with $(K, \mathcal{O}, k)$ the splitting p-modular system for G defined in Notation 36.1. We now want to look at the connections between representations of G over K (or $\mathbb{C}$), and representations of G over k.

Notation 37.1

For a complex class function $\chi \in \mathrm{Cl}(G)$, we denote the restriction of χ to G° by $\chi^\circ \in \mathrm{Cl}(G^\circ)$. We denote the set of class functions of G which vanish on $G \backslash G^\circ$ – i.e. the $\chi \in \mathrm{Cl}(G)$ for which $\chi(x) = 0$ for all $x \in G \backslash G^\circ$ – by $\mathrm{Cl}^\circ(G)$.

Corollary 37.2

Let χ be an ordinary character of G. Then χ° is a Brauer character of G.

Proof: Let X be an (ordinary) representation of G which affords the character χ. Then X is similar to a representation X' in block diagonal form, with some irreducible representations of G on the diagonal. Thus the entries in $X'(g)$ are contained in K for all $g \in G$ (see the setup in Notation 36.1). Hence by Proposition 32.2, there exists an $\mathcal{O}$-form of the KG-module corresponding to X'. In other words, X is similar to some representation of G with matrix entries in $\mathcal{O}$.

Let $X^\star$ denote the representation of G over k given by $X^\star(g) := X(g)^\star$ for all $g \in G$. Fix an element $g \in G^\circ$ and let $\xi_1, \dots, \xi_n$ be the eigenvalues of $X(g)$, where $n \in \mathbb{N}$ is the degree of X. Since $(p, o(g)) = 1$, the eigenvalues $\xi_1, \dots, \xi_n$ are p'-roots of unity so they lie in U. Then $\xi_1^\star, \dots, \xi_n^\star$ are the roots of the polynomial $\det(xI - X(g))^\star = \det(xI - X^\star(g))$, and hence $\xi_1^\star, \dots, \xi_n^\star$ are the eigenvalues of $X^\star(g)$. Therefore $X^\star$ is a representation of G over k with Brauer character χ°. ■

Corollary 37.3

The set $\mathrm{IBr}(G)$ is a basis of $\mathrm{Cl}(G^\circ)$ over $\mathbb{C}$. In particular, $|\mathrm{IBr}(G)|$ is the number of p'-conjugacy classes of G.

Proof: By Theorem 36.14, $\mathrm{IBr}(G)$ is linearly independent over $\mathbb{C}$. It remains to show that $\mathrm{IBr}(G)$ is a generating set for $\mathrm{Cl}(G^\circ)$. Let $\mu \in \mathrm{Cl}(G^\circ)$ and let $\alpha \in \mathrm{Cl}(G)$ be an extension of μ to a class function of G. Then by Proposition 35.19, since $\mathrm{Irr}(G)$ is a $\mathbb{C}$-basis for $\mathrm{Cl}(G)$,

$$\alpha = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$$

for some $a_\chi \in \mathbb{C}$, so

$$\mu = \alpha^\circ = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi^\circ.$$

By Corollary 37.2 each χ° is a Brauer character, and hence by Theorem 36.11, each χ° is a non-negative integer linear combination of irreducible Brauer characters. Therefore μ is a linear combination of irreducible Brauer characters over $\mathbb{C}$, so $\mathrm{IBr}(G)$ is a generating set for $\mathrm{Cl}(G^\circ)$. The final claim is then immediate. ∎

Remark 37.4

Let $\chi \in \mathrm{Irr}(G)$. Corollary 37.3 says that there exist positive integers $d_{\chi\varphi} \geqslant 0$ such that

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi.$$

Note that if we translate this from characters to modules, we see that the $d_{\chi\varphi}$ are the same as the decomposition numbers from Definition 24.4 and Brauer's Reciprocity, so the decomposition matrix of G with respect to p is

$$D = (d_{\chi\varphi})_{\chi \in \mathrm{Irr}(G), \varphi \in \mathrm{IBr}(G)}.$$

Corollary 37.5

The decomposition matrix D has full rank $|\mathrm{IBr}(G)|$.

Proof: Since $\mathrm{Irr}(G)$ is a basis of $\mathrm{Cl}(G)$, $\{\chi^\circ \mid \chi \in \mathrm{Irr}(G)\}$ spans $\mathrm{Cl}(G^\circ)$. There is therefore a subset $B \subseteq \{\chi^\circ \mid \chi \in \mathrm{Irr}(G)\}$ which forms a basis for $\mathrm{Cl}(G^\circ)$. By Corollary 37.3, the columns of the matrix $(d_{\chi\varphi})_{\chi \in B, \varphi \in \mathrm{IBr}(G)}$ are linearly independent. Hence D has full rank. ∎

In particular, D has no zero-columns so every $\varphi \in \mathrm{IBr}(G)$ is a constituent of at least one χ°, for some $\chi \in \mathrm{Irr}(G)$. This gives us a route to $\mathrm{IBr}(G)$, at least in principle.

Definition 37.6

The **Cartan matrix of** G (with respect to p) is defined to be

$$C := D^t D.$$

Since D has maximum rank, $C = (c_{\varphi\mu})_{\varphi,\mu \in \mathrm{IBr}(G)}$ is a positive definite symmetric matrix with non-negative integer entries. Note that for any $\varphi, \mu \in \mathrm{IBr}(G)$,

$$c_{\varphi\mu} = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi} d_{\chi\mu}.$$

Definition 37.7

Let $\varphi \in \mathrm{IBr}(G)$ be an irreducible Brauer character afforded by an irreducible k-representation X of G, and let S be a simple kG-module associated to X. Let P_S denote the projective cover of S and let Q_S denote a lift of P_S to $\mathcal{O}G$ as in Corollary 30.6. We say that the character of $KG \otimes_{\mathcal{O}G} Q_S$ is the **projective indecomposable character of** φ, and denote it by Φ_φ.

Corollary 37.8

Let $\varphi \in \mathrm{IBr}(G)$. Then

(a) $\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$, and

(b) $\Phi_\varphi^\circ = \sum_{\mu \in \mathrm{IBr}(G)} c_{\varphi\mu}\mu$.

Proof: (a) This result follows from Brauer reciprocity.

(b) This follows from part (a) because

$$\Phi_\varphi^\circ = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi^\circ = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi} \sum_{\mu \in \mathrm{IBr}(G)} d_{\chi\mu}\mu = \sum_{\mu \in \mathrm{IBr}(G)} c_{\varphi\mu}\mu$$

■

Theorem 37.9

If $p \nmid |G|$, then $\mathrm{IBr}(G) = \mathrm{Irr}(G)$ and the decomposition matrix of G is the identity matrix when the characters are ordered in the same way for the rows and for the columns.

Proof: If $p \nmid |G|$, then by Maschke's theorem, kG is semisimple. By Theorem 13.2 (c), since k is a splitting field for G, $|G| = \dim_k(kG) = \sum_{\varphi \in \mathrm{IBr}(G)} \varphi(1)^2$. We also know that $|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2$ by Exercise 35.14. Now

$$\begin{aligned} |G| &= \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = \sum_{\chi \in \mathrm{Irr}(G)} \left(\sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(1) \right)^2 = \sum_{\chi \in \mathrm{Irr}(G)} \sum_{\varphi,\mu \in \mathrm{IBr}(G)} d_{\chi\varphi}d_{\chi\mu}\varphi(1)\mu(1) \\ &\geqslant \sum_{\chi \in \mathrm{Irr}(G)} \sum_{\varphi \in \mathrm{IBr}(G)} (d_{\chi\varphi})^2\varphi(1)^2 = \sum_{\varphi \in \mathrm{IBr}(G)} \left(\sum_{\chi \in \mathrm{Irr}(G)} (d_{\chi\varphi})^2 \right) \varphi(1)^2 \geqslant \sum_{\varphi \in \mathrm{IBr}(G)} \varphi(1)^2 = |G|, \end{aligned}$$

where the last inequality follows from the fact that for every $\varphi \in \mathrm{IBr}(G)$, there is some $\chi \in \mathrm{Irr}(G)$ with $d_{\chi\varphi} \neq 0$. Hence $d_{\chi\varphi}d_{\chi\mu} = 0$ if $\varphi \neq \mu$, and for every $\varphi \in \mathrm{IBr}(G)$ there exists a unique $\chi \in \mathrm{Irr}(G)$ with $d_{\chi\varphi} \neq 0$. In fact $d_{\chi\varphi} = 1$. ■

Definition 37.10

For $\varphi, \psi \in \mathrm{Cl}(G)$ or $\mathrm{Cl}(G^\circ)$, we define

$$\langle \varphi, \psi \rangle^\circ := \frac{1}{|G|} \sum_{g \in G^\circ} \varphi(g)\overline{\psi(g)}.$$

Note that $\langle \varphi, \psi \rangle^\circ = \overline{\langle \psi, \varphi \rangle^\circ}$.

The following theorem is a replacement for the orthogonality relations from ordinary character theory.

Theorem 37.11

The set $\{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(G)\}$ is a basis for $\mathrm{Cl}^\circ(G)$. For every $\varphi, \psi \in \mathrm{IBr}(G)$ we have

$$\langle \varphi, \Phi_\psi \rangle^\circ = \delta_{\varphi\psi} = \langle \Phi_\varphi, \psi \rangle^\circ,$$

and therefore $C^{-1} = (\langle \varphi, \psi \rangle^\circ)_{\varphi,\psi \in \mathrm{IBr}(G)}$.

Proof: Let $x \in G$ and $y \in G^\circ$ and let C_x and C_y be their respective conjugacy classes in G. By the second orthogonality relations (Theorem 35.21), we have

$$\delta_{C_x,C_y}|C_G(x)| = \sum_{\chi \in \mathrm{Irr}(G)} \overline{\chi(x)}\chi(y).$$

Since $y \in G^\circ$, we know that $\chi(y) = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\varphi(y)$. Hence

$$(*) \qquad \delta_{C_x,C_y}|C_G(x)| = \sum_{\varphi \in \mathrm{IBr}(G)} \left(\sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\overline{\chi(x)} \right) \varphi(y) = \sum_{\varphi \in \mathrm{IBr}(G)} \overline{\Phi_\varphi(x)}\varphi(y)$$

by Corollary 37.8. Thus for any $x \in G \backslash G^\circ$, we have $\sum_{\varphi \in \mathrm{IBr}(G)} \overline{\Phi_\varphi(x)}\varphi = 0$. But Theorem 36.14 shows that $\mathrm{IBr}(G)$ is a linearly independent set, so $\overline{\Phi_\varphi(x)} = 0$ for all $x \in G \backslash G^\circ$ and therefore $\overline{\Phi_\varphi}$ is a class function which vanishes on $G \backslash G^\circ$, $\overline{\Phi_\varphi} \in \mathrm{Cl}^\circ(G)$.

Let $x_1, \dots, x_r$ be a system of representatives for the conjugacy classes $C_1 \dots, C_r$ in G° ($r \in \mathbb{N}$). Define the following $r \times r$ matrices.

$$\begin{aligned} I_r &:= \text{the identity } r \times r \text{ matrix} \\ \Phi &:= (\Phi_\varphi(x_i))_{\varphi \in \mathrm{IBr}(G), i=1,\dots,r}, \\ Y &:= (\varphi(x_j))_{\varphi \in \mathrm{IBr}(G), j=1,\dots,r} \\ E &:= \mathrm{diag}(|C_G(x_1)|, \dots, |C_G(x_r)|) \end{aligned}$$

Then the equation $(*)$ can be expressed as

$$I_r = \overline{\Phi}^t Y E^{-1}$$

Thus YE^{-1} is a right inverse, and hence a left inverse, for $\overline{\Phi}^t$, so

$$I_r = YE^{-1}\overline{\Phi}^t.$$

It follows that

$$\delta_{\varphi\mu} = \sum_{i=1}^{r} \varphi(x_i) \frac{1}{|C_G(x_i)|} \overline{\Phi_\mu(x_i)}.$$

Now since $\frac{|G|}{|C_i|} = |C_G(x_i)|$,

$$\langle \varphi, \Phi_\mu \rangle^\circ = \frac{1}{|G|} \sum_{g \in G^\circ} \varphi(g)\overline{\Phi_\mu(g)} = \frac{1}{|G|} \sum_{i=1}^{r} \varphi(x_i)|C_i|\overline{\Phi_\mu(x_i)} = \sum_{i=1}^{r} \varphi(x_i) \frac{1}{|C_G(x_i)|} \overline{\Phi_\mu(x_i)} = \delta_{\varphi\mu}.$$

Thus the set of projective indecomposable characters $\{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(G)\}$ is linearly independent. Since $\dim_{\mathbb{C}} \mathrm{Cl}^\circ(G) = \dim_{\mathbb{C}} \mathrm{Cl}(G^\circ) = |\mathrm{IBr}(G)|$ by Corollary 37.2, it follows that $\{\Phi_\varphi \mid \varphi \in \mathrm{IBr}(G)\}$ is a basis for $\mathrm{Cl}^\circ(G)$.

Finally, by Corollary 37.8, for any $\mu \in \mathrm{IBr}(G)$, $\Phi_\mu^\circ = \sum_{\psi \in \mathrm{IBr}(G)} c_{\psi\mu}\psi$. Thus for any $\varphi \in \mathrm{IBr}(G)$,

$$\sum_{\psi \in \mathrm{IBr}(G)} c_{\psi\mu} \langle \varphi, \psi \rangle^\circ = \langle \varphi, \sum_{\psi \in \mathrm{IBr}(G)} c_{\psi\mu}\psi \rangle^\circ = \langle \varphi, \Phi_\mu^\circ \rangle^\circ = \langle \varphi, \Phi_\mu \rangle^\circ = \delta_{\varphi\mu}.$$

Therefore $(\langle \varphi, \psi \rangle^\circ)_{\varphi,\psi \in \mathrm{IBr}(G)}$ is the inverse of the Cartan matrix C. ■