Modular Representation Theory of Finite Groups

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Contents

Chapter 10. Blocks and Defect Groups

We can break down the representation theory of finite groups into its smallest parts by studying the *blocks* of group algebras. First we will define blocks for any ring A. For the remainder of the course we will then return to the situation of a finite group G and a splitting p-modular system $(K, \mathcal{O}, k)$. We will briefly talk about the blocks of KG and $\mathcal{O}G$, and then move on to focus on the blocks of kG.

References:

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[NT89] H. Nagao and Y. Tsushima, *Representations of finite groups*, Translated from the Japanese. Academic Press, Inc., Boston, MA, 1989.

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38 Blocks

We first define blocks for any ring A with an identity.

Proposition 38.1

Let A be an arbitrary ring with an identity.

(a) The set of decompositions of A into a direct sum of two-sided ideals

$$A = A_1 \oplus \cdots \oplus A_r$$

(for some $r \in \mathbb{N}$) biject with the set of decompositions of 1_A into a sum of orthogonal central idempotents,

$$1_A = e_1 + \cdots + e_r,$$

where e_i is the identity of A_i and $A_i = Ae_i$ for $1 \leqslant i \leqslant r$.

(b) For each $1 \leqslant i \leqslant r$, the direct summand $A_i = Ae_i$ of A is indecomposable as a ring if and only if the corresponding central idempotent e_i is primitive.

(c) The decomposition of A into indecomposable two-sided ideals is unique.

Proof:

(a) Decompose A into a direct sum of two-sided ideals, $A = A_1 \oplus \cdots \oplus A_r$. Then the identity element of A decomposes into $1_A = e_1 + \cdots + e_r$, where $e_i \in A_i$ for each $1 \leqslant i \leqslant r$. For any element $a \in A$, $a = 1_A a = (e_1 + \cdots + e_r)a = e_1 a + \cdots + e_r a$, with $e_i a \in A_i$. If $a_i \in A_i$ then $e_i a_i = a_i$ and $e_j a_i = 0$ for $j \neq i$. In other words, e_i is the identity of A_i and $e_i^2 = e_i$ so $\{e_i\}_{i=1}^r$ is a set of orthogonal central idempotents of A, and $A_i = Ae_i$ for $1 \leqslant i \leqslant r$.

Conversely, if $\{e_i\}_{i=1}^r$ is a set of central orthogonal idempotents of A such that $1_A = \sum_{i=1}^r e_i$, then $\bigoplus_{i=1}^r Ae_i$ is a direct sum decomposition of A into two-sided ideals.

(b) $\Rightarrow$ Suppose that $A_i = Ae_i$ and $e_i = f + j$ for orthogonal idempotents f, j. Then $A_i = Af \oplus Aj$, where the sum is direct because if $a \in Af \cap Aj$ then $a = af$ and $a = aj$, hence $a = afj = 0$. Thus, if e_i is not primitive, then A_i is not indecomposable.

$\Leftarrow$ Now suppose that $A_i = Ae_i$ and $A_i = L_1 \oplus L_2$ for two two-sided ideals L_1 and L_2 of A. Then $e_i = f + j$ for some $f \in L_1$, $j \in L_2$. Since $L_1 \cap L_2 = \{0\}$ and $fj \in L_1 \cap L_2$, we have $fj = 0$. As e_i is the identity in A_i, we also see that $f = e_i f = (f + j)f = f^2 + jf = f^2$ and similarly, $j^2 = j$, hence f and j are orthogonal idempotents, so e_i is not primitive.

(c) Finally, suppose that $A = A_1 \oplus \cdots \oplus A_r$ for some $r \in \mathbb{N}$, and suppose that L is an indecomposable direct summand of A from a different decomposition of A. Every $x \in L$ has a decomposition $x = a_1 + \cdots + a_r$ with $a_i \in A_i$ for all $1 \leqslant i \leqslant r$. Then $e_i x = a_i$, and this is an element of L because L is a two-sided ideal. Hence $L = (L \cap A_1) + \cdots + (L \cap A_r)$. This is a decomposition of L, which was indecomposable, hence $L = L \cap A_m$ for some $1 \leqslant m \leqslant r$. By the indecomposability of A_m, this must be the whole of A_m. Thus the decomposition of A into indecomposable two-sided ideals is unique. ■

Definition 38.2

Let $A = A_1 \oplus \cdots \oplus A_r$ be the unique decomposition of A into a direct sum of indecomposable two-sided ideals such that $A_i = Ae_i$ for $1 \leqslant i \leqslant r$, as above. Each A_i is a **block** of A and the corresponding primitive central idempotent e_i is called the **block idempotent of** A_i. Note that the blocks of A are direct summands of A, and therefore are projective as A-modules.

Definition 38.3

Let M be an A-module. Then M **lies in the block** $A_i = Ae_i$ if $e_i M = M$ and $e_j M = 0$ for all $j \neq i$.

Proposition 38.4

Let M be an A-module. Then M has a unique direct sum decomposition $M = M_1 \oplus \cdots \oplus M_r$ where M_i lies in the block A_i for $1 \leqslant i \leqslant r$. In particular, every indecomposable A-module lies in a uniquely determined block of A.

Proof: Let $m \in M$. Then $m = 1_A m = e_1 m + \cdots + e_r m$, so $M = e_1 M + \cdots + e_r M$. Denote $e_i M$ by M_i for $1 \leqslant i \leqslant r$. Suppose that $x \in M_i \cap M_j$ for some $i \neq j$. Then $x = e_i x$ and $x = e_j x$ (because the e_n's act as the identity on their respective blocks for $1 \leqslant n \leqslant r$), so $x = e_i(e_j x) = 0$. Thus the sum is direct. Moreover, $e_i M_i = e_i e_i M = e_i M = M_i$ and $e_j M_i = 0$ for all $j \neq i$, so M_i lies in the block A_i for each $1 \leqslant i \leqslant r$.

Suppose that $M = N_1 \oplus \cdots \oplus N_r$ is another direct sum decomposition of M with N_i in block A_i for $1 \leqslant i \leqslant r$. Then $N_i = e_i N_i \subseteq e_i M = M_i$ (because $N_i \subseteq M$) and hence (since N_i and M_i are indecomposable), $N_i = M_i$ for each $1 \leqslant i \leqslant r$.

The final claim follows immediately from the first. ■

Corollary 38.5

Suppose that an A-module M lies in the block A_i. Then every submodule and factor module of M lies in A_i.

Proof: Let $V \subseteq M$ be a submodule of M. Then for $i \neq j$, $e_j V \subseteq e_j M = 0$ so V must lie in A_i. We also have $e_j(M/V) \subseteq e_j M/e_j V = 0$ so M/V also lies in A_i. ■

Definition 38.6

Let X, Y and Z be A-modules. We say that X is **a non-split extension of Y by Z** iff there exists a non-split exact sequence $0 \to Y \to X \to Z \to 0$.

The following theorem characterises when two modules are in the same block for an algebra over a field.

Theorem 38.7

Let A be an algebra over a field. Let S, T be simple A-modules. The following are equivalent.

(a) S and T lie in the same block of A.

(b) There exist simple A-modules $S = S_1, \dots, S_m = T$ such that S_i, S_{i+1} are composition factors of the same projective indecomposable A-module for $1 \leqslant i \leqslant m-1$.

(c) There exist simple A-modules $S = S_1, \dots, S_m = T$ such that there exists a non-split extension of S_i by S_{i+1} (or vice versa) for $1 \leqslant i < m$.

Proof: Omitted. ■

We now return to the notation of the previous chapter, with G a finite group and $(K, \mathcal{O}, k)$ the splitting p-modular system for G given in Notation 36.1.

Remark 38.8 (*Blocks of KG*)

Recall that KG is semisimple since K is of characteristic 0. It follows from Scholium 12.6, Proposition 35.17 and Definition 38.2 that

$$KG = \bigoplus_{\chi \in \mathrm{Irr}(G)} KGe_\chi.$$

In other words, the blocks of KG are labelled by the ordinary irreducible characters $\chi \in \mathrm{Irr}(G)$, and e_χ is the block idempotent for the block KGe_χ.

Remark 38.9 (*Blocks of $\mathcal{O}G$ and kG*)

We know from Proposition 30.3 (d) that there is a bijection between the central idempotents of $\mathcal{O}G$ and the central idempotents of kG. Hence a decomposition $1_{\mathcal{O}G} = b_1 + \cdots + b_r$ of the identity element of $\mathcal{O}G$ into a sum of central primitive idempotents of $\mathcal{O}G$ corresponds to a decomposition $1_{kG} = \overline{b}_1 + \cdots + \overline{b}_r$ of the identity element of kG into a sum of central primitive idempotents of kG. In other words, there is a bijection between the blocks of $\mathcal{O}G$ and the blocks of kG. Note that these are the blocks we are interested in! We sometimes refer to the blocks of kG as the *p-blocks of G*.

Remark 38.10

Let S be a simple kG-module. If S is in a block B of kG then the projective cover P_S of S is also in B, by Corollary 38.5.

Example 38.11

Let G be a p-group. Then kG has exactly one simple module by Corollary 17.3. Hence by Theroem 38.7, all indecomposable modules lie in the same block, so kG has just one block.

39 Defect Groups

We continue with the splitting p-modular system $(K, \mathcal{O}, k)$ for G. From now on we will only discuss the blocks of kG. Analogous results hold for the corresponding blocks of $\mathcal{O}G$. In this section we will study an important block invariant: the defect group.

> **Top Tip:** It will be helpful to recall the definition of the vertex of a module (Defintion 28.2) before going any further!

Remark 39.1

The blocks of kG can be viewed as indecomposable modules of $k[G \times G]$. First of all, notice that kG is a $k[G \times G]$-module with the action of $G \times G$ on kG given by

$$\begin{aligned}(G \times G) \times kG &\to kG \\ ((g_1, g_2), a) &\mapsto g_1 a g_2^{-1},\end{aligned}$$

linearly extended to an action of $k[G \times G]$. A two-sided ideal of kG is, by definition, a submodule of kG which is closed under left and right multiplication by elements of G. In other words, it is a submodule of kG closed under the action of $G \times G$ as defined above – i.e. a $k[G \times G]$-submodule of kG. Thus a block of kG can be viewed as an indecomposable $k[G \times G]$-submodule of kG considered as a $k[G \times G]$-module.

Notation 39.2

Denote the diagonal embedding of G in $G \times G$ by

$$\begin{aligned}\delta : G &\to G \times G \\ g &\mapsto (g, g).\end{aligned}$$

Theorem 39.3

Let B be a block of kG. Every vertex of B, considered as an indecomposable $k[G \times G]$-module, has the form $\delta(D)$ for a p-subgroup $D \leqslant G$. The group D is uniquely determined up to conjugation in G.

Proof: First we show that the $k[G \times G]$-module B is relatively $\delta(G)$-projective. Since B is a direct summand of kG, it is enough to show that kG is $\delta(G)$-projective. But kG contains the subspace $k.1$, which is the trivial $k\delta(G)$-module. Further, when we consider the dimension of these k-vector spaces we see that

$$\dim_k(kG) = |G| \dim_k(k.1) = |G \times G : \delta(G)| \dim_k(k.1).$$

By arguments in Remark 20.7,

$$\dim_k(k.1)\uparrow_{\delta(G)}^{G\times G} = |G\times G : \delta(G)| \dim_k(k.1).$$

Consider the homomorphism $\varphi : k.1 \to kG$ sending $k.1$ to $k.1$. By Proposition 20.8 (the universal property of induction) there then exists a $k[G\times G]$-homomorphism

$$\overline{\varphi} : (k.1)\uparrow_{\delta(G)}^{G\times G} \to kG.$$

Since kG is generated by $k.1$, $\overline{\varphi}$ is surjective. Since the two modules have the same dimension, $\overline{\varphi}$ is an isomorphism and hence $kG \cong (k.1)\uparrow_{\delta(G)}^{G\times G}$. It follows that kG, and therefore B, is $\delta(G)$-projective.

By Definition 28.2, then a vertex of B (still considered as a $k[G\times G]$-module) lies in $\delta(G)$. This vertex is a p-group (Proposition 28.4), and thus is the image $\delta(D)$ of some p-subgroup of G, showing the first part.

We know that $\delta(D)$ is uniquely determined up to conjugacy in $G\times G$. We want to show that D this is unique up to conjugation by elements of G. Suppose that D' is another p-subgroup of G such that $\delta(D')$ is a vertex of B. Then $\delta(D') = {}^{(g_1,g_2)}\delta(D)$ for some $(g_1,g_2) \in G\times G$. If $x \in D$, then ${}^{(g_1,g_2)}(x,x) = ({}^{g_1}x, {}^{g_2}x) \in \delta(D')$. Hence ${}^{g_1}x \in D'$ for all $x \in D$. Since D and D' have the same order, it follows that ${}^{g_1}D = D'$. In particular, D is uniquely determined up to conjugation by elements of G. ■

Definition 39.4

Let B be a block of kG. A **defect group** of B is a p-subgroup D of G such that $\delta(D)$ is a vertex of B considered as a $k[G\times G]$-module. The defect group of a block is uniquely determined up to G-conjugacy. If a defect group D of B has order p^d then d is called the **defect** of B.

Why are defect groups useful and important? We will shortly see that they measure how far a block is from being semisimple.

Lemma 39.5

Let B be a block of kG with defect group D. Then B is relatively D-projective when thought of as a kG-module via conjugation.

Proof: By definition B is a projective kG-module with the usual module structure given by left multiplication. We can also think of B as a kG-module by linearly extending the conjugation action, $g.x = gxg^{-1}$ for all $x \in B$ and all $g \in G$. Then since $G \cong \delta(G)$, we can define a $k[\delta(G)]$-module structure on B via $(g,g).x = gxg^{-1}$ for all $x \in B$, $(g,g) \in \delta(G)$. Notice that this $k[\delta(G)]$-module is just a restriction of the $k[G\times G]$-module, $B\downarrow_{\delta(G)}^{G\times G}$. We will show that B is relatively $\delta(D)$-projective as a $k[\delta(G)]$-module, and hence via the isomorphism above, B is relatively D-projective when thought of as a kG-module via conjugation.

By the definition of defect groups, $\delta(D)$ is a vertex of B as a $k[G\times G]$-module. Thus B is a direct summand of $V\uparrow_{\delta(D)}^{G\times G}$ for some $k[\delta(D)]$-module V. Hence by restricting to $\delta(G)$ and applying the Mackey formula, we have

$$B\downarrow_{\delta(G)}^{G\times G} \mid V\uparrow_{\delta(D)}^{G\times G}\downarrow_{\delta(G)}^{G\times G} \cong \bigoplus_{(g_1,g_2)\in[\delta(G)\backslash G\times G/\delta(D)]} \left({}^{(g_1,g_2)}V\downarrow_{\delta(G)\cap {}^{(g_1,g_2)}\delta(D)}^{{}^{(g_1,g_2)}\delta(D)}\right)\uparrow_{\delta(G)\cap {}^{(g_1,g_2)}\delta(D)}^{\delta(G)}.$$

Therefore B, considered as a $k[\delta(G)]$-module, is a sum of induced $k[\delta(G)\cap^{(g_1,g_2)}\delta(D)]$-modules for some $g_1,g_2 \in G$, and hence is relatively $\delta(G)\cap^{(g_1,g_2)}\delta(D)$-projective for some $g_1,g_2 \in G$. It is now enough to show that these groups are conjugate in $\delta(G)$ to a subgroup of $\delta(D)$ as this would imply that B is then relatively $\delta(D)$-projective, as required.

Let $(g_1, g_2) \in G \times G$. Any element of $\delta(G) \cap {}^{(g_1,g_2)} \delta(D)$ is of the form ${}^{(g_1,g_2)}\delta(d) = {}^{(g_1,g_2)}(d, d)$ for some $d \in D$ such that $g_1 d g_1^{-1} = g_2 d g_2^{-1}$. Therefore ${}^{(g_1,g_2)}\delta(d) = (g_1 d g_1, g_1 d g_1^{-1}) = \delta(g_1)\delta(d)\delta(g_1)^{-1}$ and this is an element of $\delta(g_1)\delta(D)\delta(g_1)^{-1}$. It follows that $\delta(G) \cap^{(g_1,g_2)} \delta(D)$ is conjugate in $\delta(G)$ to a subgroup of $\delta(D)$. ■

Theorem 39.6

Let B be a block of kG with defect group D. Then every indecomposable kG-module in B is relatively D-projective, and hence has a vertex in D.

Proof: As in the previous lemma, we consider B as a kG-module via the conjugation action. Let V be a kG-module with the usual action of G via left multiplication. We define a linear map

$$\begin{aligned} \phi : B \otimes V &\to V \\ x \otimes v &\mapsto xv. \end{aligned}$$

Since $g.(x \otimes v) = gxg^{-1} \otimes gv$ and $\phi(g.(x \otimes v)) = gxg^{-1}gv = gxv = g.(xv)$ for all $x \in B$, $v \in V$ and $g \in G$, the map ϕ is a kG-homomorphism.

On the other hand, let b be the block idempotent of B and define another linear map

$$\begin{aligned} \psi : V &\to B \otimes V \\ v &\mapsto b \otimes v. \end{aligned}$$

For any $g \in G$ and $v \in V$, $\psi(g.v) = b \otimes gv$. But b is central in B and we are considering B as a kG-module via conjugation, so $g.b = b$ and therefore $\psi(g.v) = g.(b \otimes v)$ for all $g \in G$ and $v \in V$. Thus ψ is also a kG-homomorphism.

If the module V lies in the block B, then for any $v \in V$,

$$\phi \circ \psi(v) = \phi(b \otimes v) = bv = v,$$

so $\phi \circ \psi$ is the identity map on V. Therefore ϕ is surjective and ψ is injective, and hence $B \otimes V \cong V \oplus \ker(\phi)$.

In Lemma 39.5 we showed that B is relatively D-projective. It then follows from Exercise 31 (a) that $B \otimes V$ is also relatively D-projective, and hence, V is D-projective. In particular, every indecomposable kG-module in B has a vertex in D. ■

Corollary 39.7

Let B be a block of kG with trivial defect groups. Then B is a simple algebra, and in particular, is semisimple.

Proof: If B has trivial defect group $D = 1$, then by Theorem 39.6 every indecomposable kG-module in B is 1-projective, and hence projective. Thus every submodule of a B-module is a direct summand of that B-module. Hence B is semisimple as all its modules are semisimple. But B is an indecomposable algebra by definition. Hence B is simple. ■

We will see later that the converse of this Corollary is also true.

Finally we come to the main theorem of this section. It shows that defect groups are far from arbitrary: defect groups contain every normal p-subgroup of G, and a defect group is a radical p-subgroup of G.

Definition 39.8

Let Q be a p-subgroup of G. If Q is the largest normal p-subgroup of $N_G(Q)$ (i.e. $Q = O_p(N_G(Q))$, then Q is a **radical p-subgroup of G**.

Theorem 39.9

Let B be a block of kG with defect group D.

(a) D contains every normal p-subgroup of G.

(b) D is a radical p-subgroup of G.

Proof: Omitted. ■

Example 39.10

Let G be a p-group. We already saw that kG has only one block. Then Theorem 39.9 shows that this block has defect group $D = G$.

40 Brauer's Main Theorems

Definition 40.1

Let $H \leqslant G$, let b be a block of kH and let B be a block of kG. Then the block B **corresponds to** b if and only if b, as a $k[H \times H]$-module, is a direct summand of the restriction $B{\downarrow}^{G\times G}_{H\times H}$, *and* B is the unique block of kG with this property. We then write $B = b^G$. If such a B exists, then we say the b^G **is defined**.

We will need the following technical result for the proofs that follow.

Remark 40.2

Let $H \leqslant G$,and let $Q \leqslant H$ be a p-subgroup such that $C_G(Q) \leqslant H$. Note that the restriction $kG{\downarrow}^{G\times G}_{H\times H}$ is a disjoint union of the double H-H-cosets,

$$kG{\downarrow}^{G\times G}_{H\times H} = \bigoplus_{t\in[H\backslash G/H]} kHtH = kH \oplus \bigoplus_{t\in[H\backslash G/H],t\notin H} kHtH.$$

Fact: if $t \notin H$ then the $k[H \times H]$-submodule $kHtH$ of kG has no direct summands with vertex containing $\delta(Q)$.

In particular, if X is an indecomposable direct summand of $kG{\downarrow}^{G\times G}_{H\times H}$ with vertex containing $\delta(Q)$, then X is a direct summand of kH, so X is a block of kH.

Proposition 40.3 (*Facts about b^G*)

Let $H \leqslant G$ and let b be a block of kH with defect group D.

(a) If b^G is defined, then D lies in a defect group of b^G.

(b) If $H \leqslant N \leqslant G$, and b^N, $(b^N)^G$ and b^G are defined, then $b^G = (b^N)^G$.

(c) If $C_G(D) \leqslant H$ then b^G is defined.

Proof: (a) Let $B := b^G$ and let E be a defect group of B. By the definition of defect groups, $\delta(E)$ is a vertex of the $k[G \times G]$-module B, so B is a direct summand of $V\uparrow_{\delta(E)}^{G\times G}$ for some $\delta(E)$-module V. Since b is a direct summand of $B\downarrow_{H\times H}^{G\times G}$, it follows from the Mackey formula that b is a direct summand of

$$V\uparrow_{\delta(E)}^{G\times G}\downarrow_{H\times H}^{G\times G} = \bigoplus_{x\in[H\times H\backslash G\times G/\delta(E)]} \left(\left({}^{x}V\downarrow^{{}^{x}\delta(E)}_{(H\times H)\cap {}^{x}\delta(E)}\right)\uparrow^{H\times H}_{(H\times H)\cap {}^{x}\delta(E)}\right).$$

Hence b is a direct summand of a module induced from $(H \times H) \cap {}^{x}\delta(E)$, for some $x \in G \times G$. In particular, b is a direct summand of a module induced from a conjugate of a subgroup of $\delta(E)$. Since b has defect group D, $\delta(D)$ is a vertex of b so $\delta(D)$ is minimal such that b is relatively $\delta(D)$-projective. It follows that $\delta(D)$ is conjugate to a subgroup of $\delta(E)$.
Suppose that $(g_1, g_2)\delta(D)(g_1, g_2)^{-1} \leqslant \delta(E)$. Then $g_1 D g_1^{-1} \leqslant E$ and hence $D \leqslant g_1^{-1} E g_1$, which is a defect group of B, showing part (a).

(b) Part (b) follows from the definitions. Since b^N is defined, b is a direct summand of $b^N\downarrow_{H\times H}^{N\times N}$, and b^N is the unique such block. Since $(b^N)^G$ is defined, b^N is a direct summand of $(b^N)^G\downarrow_{N\times N}^{G\times G}$ and $(b^N)^G$ is the unique such block. Therefore b is a direct summand of $b^N\downarrow_{H\times H}^{N\times N}$ which is a direct summand of $(b^N)^G\downarrow_{H\times H}^{N\times N}\downarrow_{N\times N}^{G\times G} = (b^N)^G\downarrow_{H\times H}^{G\times G}$. However, since b^G is defined, b is also a direct summand of $b^G\downarrow_{H\times H}^{G\times G}$, and b^G is the unique such block. Therefore $b^G = (b^N)^G$.

(c) Suppose now that $C_G(D) \leqslant H$. To prove part (c), it is enough to show that b occurs precisely once in a decomposition of $kG\downarrow_{H\times H}^{G\times G}$ into indecomposable modules, as then there is a unique indecomposable direct summand of kG (i.e. a block of kG) such that b is a direct summand of the restriction of that summand to $H \times H$.
As in Remark 40.2,

$$kG\downarrow_{H\times H}^{G\times G} = kH \oplus \bigoplus_{t\in[H\backslash G/H] t\notin H} kHtH.$$

Now kH is, as a $k[H \times H]$-module, a direct sum of blocks of kH, which are not isomorphic to each other. Thus b is a direct summand of kH with multiplicity one. But b has vertex $\delta(D)$. Remark 40.2 shows that if $t \notin H$ then no direct summand of $kHtH$ has a vertex containing $\delta(D)$. Thus b is not a direct summand of any $kHtH$ for $t \notin H$, so b has multiplicity one in $kG\downarrow_{H\times H}^{G\times G}$, as required. ■

We now prove a special case of Brauer's first main theorem. The result holds true for any subgroup N of G containing $N_G(D)$, but we will only consider the case where $N = N_G(D)$ as this situation gives rise to the **Brauer correspondence**.

Theorem 40.4 (*Brauer's First Main Theorem*)

Let $D \leqslant G$ be a p-subgroup and let $N := N_G(D)$. Then $b \mapsto b^G$ defines a bijection between the blocks of kN with defect group D, and the blocks of kG with defect group D.

$$\{\text{ Blocks of } kN \text{ with defect group } D\} \to \{\text{ Blocks of } kG \text{ with defect group } D\}$$
$$b \mapsto b^G$$

In this case we call b^G **the Brauer correspondent** of b.

Proof: Let b be a block of kN with defect group D. Then $\delta(D)$ is a vertex of b considered as a $k[N \times N]$-module. The Green correspondence (Theorem 29.4) shows that there exists a unique indecomposable direct summand $g(b)$ of $b\uparrow_{N\times N}^{G\times G}$ with vertex $\delta(D)$. Moreover, by the proof of part (b) of the Green Correspondence, b occurs as a direct summand of $g(b)\downarrow_{N\times N}^{G\times G}$ with multiplicity one.

By Proposition 40.3 (c), the block b^G is defined, and hence b^G is the unique indecomposable $k[G\times G]$-module such that b occurs in its restriction to $N \times N$. Therefore $b^G \cong g(b)$ so b^G has vertex $\delta(D)$ when considered as a $k[G \times G]$-module. In particular, b^G has defect group D and so $b \mapsto b^G$ is an injective map from blocks of kN with defect group D, to blocks of kG with defect group D.

We now show that this map is surjective. Suppose B is a block of kG with defect group D. Then B is an indecomposable $k[G \times G]$-module with vertex $\delta(D)$ and the Green correspondence shows that $B\downarrow_{N\times N}^{G\times G}$ has a unique direct summand, $f(B)$, with vertex $\delta(D)$. As B is a direct summand of kG, $B\downarrow_{N\times N}^{G\times G}$ and hence $f(B)$, is a direct summand of $kG\downarrow_{N\times N}^{G\times G}$. In Remark 40.2 we saw that any direct summand of $kG\downarrow_{N\times N}^{G\times G}$ with vertex containing $\delta(D)$ is an indecomposable direct summand of kN. Therefore $f(B)$ is a block of kN with vertex $\delta(D)$ when considered as a $k[N \times N]$-module, so $f(B)$ is a block of kN with defect group D. It follows from part (c) of the Green Correspondence that $B \cong g(f(B))$, and $g(f(B)) \cong f(B)^G$ by the first part of the proof, so the map $b \mapsto b^G$ is surjective. ■

Theorem 40.5 (*Brauer's Second Main Theorem*)

Let $H \leqslant G$, let B be a block of kG and let b be a block of kH. Suppose that V is an indecomposable module in B and U is an indecomposable module in b with vertex Q such that $C_G(Q) \leqslant H$. If U is a direct summand of $V\downarrow_H^G$, then b^G is defined and $b^G = B$.

Proof: First we note that Theorem 39.6 shows that there is a defect group D of b which contains the vertex Q of U. Hence $C_G(D) \leqslant C_G(Q)$, which is contained in H by assumption. Thus by Proposition 40.3 (c), b^G is defined.

Suppose that $B \neq b^G$. Let e be the block idempotent of b so $b = kHe = ekH$. For a kH-module X, Proposition 38.4 shows that there is a decomposition $X = eX \oplus (1-e)X$, where eX lies in b and $(1-e)X$ does not. If $X = Y \oplus Z$ then $eX = eY \oplus eZ$ is still a direct sum. Applying this to the decomposition of kG as a $k[H \times H]$-module as in Remark 40.2, if we fix

$$M := \bigoplus_{t \in [H\backslash G/H] t \notin H} kHtH$$

then

$$ekG = ekH \oplus eM = b \oplus eM$$

as kH-modules. But kG, b and M are all $k[H \times H]$-modules, and e commutes with all elements of H, so we also have $ekG = b \oplus eM$ as $k[H \times H]$-modules.

Now since B is a direct summand of kG, eB is a direct summand of $b \oplus eM$ as $k[H \times H]$-modules. But b is an indecomposable $k[H \times H]$-module and is not a direct summand of $B\downarrow_{H\times H}^{G\times G}$ by assumption, so eB is a direct summand of eM, which is a direct summand of M as $k[H \times H]$-modules. Hence by Remark 40.2, no direct summand of the $k[H \times H]$-module eB has a vertex containing $\delta(Q)$.

By the Mackey formula, the direct summands of $eB\downarrow^{H\times H}_{\delta(H)}$ are induced from subgroups of the form $\delta(H)\cap {}^{x}T$, where T is a vertex of an indecomposable summand of eB and $x\in H\times H$. It follows that no direct summand of the $k[\delta(H)]$-module $eB\downarrow^{H\times H}_{\delta(H)}$ has a vertex containing $\delta(Q)$. By transport of structure via the isomorphism $H\cong\delta(H)$, we see that eB, considered as a kH-module by conjugation (i.e. with the action of H given by $h.ex = hexh^{-1}$, for all $h\in H$, $x\in B$), has no direct summands with vertex containing Q. Hence by Exercise 31 (a), $eB\otimes eV$ also has no direct summands with vertex containing Q. Now since U is a module in b, $eU=U$. By assumption, U is a direct summand of $V\downarrow^G_H$ and this implies that $eU=U$ is a direct summand of eV. Recalling that U has vertex Q, it is then enough to show that eV is a summand of $eB\otimes eV$ as this will give us a contradiction.

Define a map

$$\begin{aligned}\varphi: eV &\to eB\otimes eV\\ v&\mapsto ef\otimes v,\end{aligned}$$

where f is the block idempotent of B (so $B=kGf$). Then for any $h\in H$,

$$\varphi(hv)=ef\otimes hv = hefh^{-1}\otimes hv = h(ef\otimes v)=h\varphi(v),$$

so φ is a kH-homomorphism. Define a second map,

$$\begin{aligned}\psi: eB\otimes eV &\to eV\\ a\otimes v&\mapsto av.\end{aligned}$$

This is also a kH-homomorhpism – for all $h\in H$,

$$\psi(h(a\otimes v))=\psi(hah^{-1}\otimes hv)=hah^{-1}hv=h(av)=h(\psi(a\otimes v)).$$

Now since V is a module in B, for any $v\in eV$ we have

$$\psi(\varphi(v))=\psi(ef\otimes v)=efv=ev=v.$$

Hence φ is injective and ψ is surjective and therefore eV is a direct summand of $eB\otimes eV$, as required to give a contradiction. ■

Lemma 40.6

Let S be a simple kG-module. Then $O_p(G)$, the largest normal p-subgroup of G, acts trivially on S. In particular, the simple kG-modules are precisely the $k[G/O_p(G)]$-modules made into kG-modules via the quotient homomorphism $G\to G/O_p(G)$.

Proof: Let $P=O_p(G)$ be the largest normal p-subgroup of G and let S be a simple kG-module. Suppose that W is a simple kP-submodule of S. Then W is the trivial kP-module because P is a p-group. Let

$$C_S(P):=\{s\in S\mid ps=s \text{ for all } p\in P\}.$$

We have $W\leqslant C_S(P)$ so $C_S(P)\neq 0$. But P is normal in G, so $C_S(P)$ is a kG-submodule of S, which was simple, and hence $C_S(P)=S$. In other words, P acts trivially on S. The final claim follows immediately. ■

Corollary 40.7

Let B be a block of kG with defect group D. Then there exists an indecomposable kG-module in B with vertex D.

Proof: Let b be a block of $N := N_G(D)$ with defect group D such that B is the Brauer correspondent of b, as defined in Brauer's First Main Theorem 40.4. As D is a defect group of b, $D = O_p(N)$ by Theorem 39.9 (b). Let S be a simple kN-module in b. It follows from Lemma 40.6 that D acts trivially on S and so S can be thought of as a simple $k[N/D]$-module. Let P_S be the projective cover of S (so P is a $k[N/D]$-module). Corollary 38.5 shows that P_S is also in the block b. We will show that P_S has vertex D and that the Green correspondent of P_S is an indecomposable kG-module in B with vertex D.

Denote the trivial kD-module by k. The module P_S is an indecomposable projective $k[N/D]$-module, so it is a direct summand of the free module $k[N/D] \cong k\uparrow_D^N$. Hence P_S is relatively D-projective. Since $D \trianglelefteq N$, it follows from Clifford's Theorem (Theorem 23.2) that $k\uparrow_D^N\downarrow_D^N$ is a direct sum of N-conjugates of k. In other words $P_S\downarrow_D^N$ is a direct summand of $k\uparrow_D^N\downarrow_D^N$, which is a direct sum of copies of the trivial kD-module k. The vertex of the trivial kD-module k is a Sylow p-subgroup of D, by Proposition 28.4 (c), and thus is equal to D. Therefore the direct summands of $P_S\downarrow_D^N$ all have vertex D. By Exercise 32, however, $P_S\downarrow_D^N$ has at least one direct summand with the same vertex as P_S. Hence D is a vertex of P_S.

Now consider P_S as an indecomposable $k[N \times N]$-module and let V be the indecomposable $k[G \times G]$-module with vertex D which is the Green correspondent of P_S. Then by Brauer's First Main Theorem 40.4, V lies in B, so B contains an indecomposable kG-module with vertex D. ■

Our final result shows that the opposite direction of Corollary 39.7 also holds.

Corollary 40.8

A block B of kG is a simple algebra if and only if B has trivial defect groups.

Proof: If B is a block of kG with trivial defect groups then B is a simple algebra by Corollary 39.7.

Suppose now that B is a block of kG which is a simple algebra. Then B is semisimple so all B-modules are projective. Hence all indecomposable B-modules have trivial vertices so by Corollary 40.7, B has trivial defect groups. ■