Modular Representation Theory of Finite Groups

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Contents

Chapter 7. The Green Correspondence

The goal of this chapter is to prove Green's correspondence. First we will need to generalise the idea of projective modules seen in Chapter 6 by defining what is called **relative projectivity**. In the second section we will define **vertices** and **sources** of indecomposable modules. Finally, in the third section we will state and prove **Green's correspondence**.

Notation: throughout this chapter, unless otherwise specified, we let G denote a finite group and let K denote a field of characteristic p. All modules over group algebras considered are assumed to be **finitely generated**, hence **of finite K-dimension**.

References:

[Web16] P. WEBB, *A course in finite group representation theory*, Cambridge Studies in Advanced Mathematics, vol. 161, Cambridge University Press, Cambridge, 2016.

[Alp86] J. L. ALPERIN, *Local representation theory*, Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986.

27 Relative Projectivity

Relative projectivity is a refinement of the idea of projectivity seen in Chapter 6. A projective module is a summand of a free module. For $H \leqslant G$, we will first define H-free modules. Then a module is relatively H-projective if it is a summand of an H-free module. Relative projectivity enables us to explore the relationship between representations of a group and representations of its subgroups. This is a very important tool for modular representation theory.

Definition 27.1

Let $H \leqslant G$. A KG-module is **H-free** if it is of the form $V{\uparrow}_H^G$ for some KH-module V. A KG-module is **relatively H-projective**, or **H-projective**, if it is isomorphic to a direct summand of an H-free module – that is, if it is isomorphic to a direct summand of a module of the form $V{\uparrow}_H^G$ for some KH-module V.

Remark 27.2

- Free $\iff$ $\{1\}$-free: a free KG-module is of the form $(KG^\circ)^n$ for some $n \in \mathbb{N}$. But $KG^\circ \cong K{\uparrow}_{\{1\}}^G$ (Chapter 4, Ex. 11) so $(KG^\circ)^n \cong (K^n){\uparrow}_{\{1\}}^G$. Hence being free and $\{1\}$-free is the same.

Therefore H-freeness is a generalisation of freeness.

- Projective $\iff$ $\{1\}$-projective: A KG-module is projective $\Leftrightarrow$ it is a summand of a free module $\Leftrightarrow$ it is a summand of a $\{1\}$-free module $\Leftrightarrow$ it is relatively $\{1\}$-projective. Therefore relative projectivity is a generalisation of projectivity.

Exercise 27.3 (*Relative freeness*)

Let $H \leqslant G$. Suppose that V is a relatively H-free KG-module with respect to a KH-submodule X, and suppose that W is a relatively H-free KG-module with respect to a KH-submodule Y. Prove that if $X \cong Y$ as KH-modules, then $V \cong W$ as KG-modules.

Exercise 27.4 (*Relative projectivity*)

Let $H \leqslant J \leqslant G$. Let U be a KG-module and let V be a KJ-module. Prove the following statements.

(a) If U is H-projective then U is J-projective.

(b) If U is a summand of $V\uparrow_J^G$ and V is H-projective, then U is H-projective.

(c) For any $g \in G$, U is H-projective if and only if ${}^gU$ is ${}^gH$-projective.

Notation 27.5 (*Induction and restriction of homomorphisms*)

Let $H \leqslant G$. Let $\varphi : U_1 \to U_2$ be a KH-homomorphism. Then the induced KG-homomorhpism

$$\begin{aligned} \mathrm{Id}_{KG} \otimes \varphi : U_1\uparrow_H^G &\to U_2\uparrow_H^G \\ g \otimes u &\mapsto g \otimes \varphi(u). \end{aligned}$$

is denoted by $\varphi\uparrow_H^G$.

On the other hand, since a KG-homomorhpism $\psi : V_1 \to V_2$ is also a KH-homomorhpism $V_1\downarrow_H^G \to V_2\downarrow_H^G$, we just denote the KH-homomorphism by ψ again, without any arrows.

The following notation will be needed in the proof of the next proposition about characterisations of relative projectivity.

Notation 27.6 (*The μ and ϵ maps*)

Let $H \leqslant G$. Let U be a KH-module and recall that $U\uparrow_H^G = \bigoplus_{g\in[G/H]} g \otimes U$ (see Remark 20.7), and restricting back to H gives a KH-module $U\uparrow_H^G\downarrow_H^G = \bigoplus_{g\in[G/H]} g \otimes U$. We denote the inclusion map from U onto the summand with $g = 1$ by μ:

$$\begin{aligned} \mu : U &\to U\uparrow_H^G\downarrow_H^G \\ u &\mapsto 1 \otimes u. \end{aligned}$$

Let V be a KG-module. Then $V\downarrow_H^G\uparrow_H^G = \bigoplus_{g\in[G/H]} g\otimes(V\downarrow_H^G)$ and we define a KG-module homorphism ϵ as follows.

$$\begin{aligned} \epsilon : V\downarrow_H^G\uparrow_H^G &\to V \\ g \otimes v &\mapsto gv \end{aligned}$$

for $g \in G$ and $v \in V\downarrow_H^G$. Note that for any $u \in U$, $\epsilon \circ \mu(u) = \epsilon(1 \otimes u) = u$, so μ is a KH-section for ϵ.

Now consider the following maps:

$$\begin{aligned}\Phi : \mathrm{Hom}_{KG}(U{\uparrow}_H^G, V) &\to \mathrm{Hom}_{KH}(U, V{\downarrow}_H^G)\\ \psi &\mapsto \psi \circ \mu\\ \Psi : \mathrm{Hom}_{KH}(U, V{\downarrow}_H^G) &\to \mathrm{Hom}_{KG}(U{\uparrow}_H^G, V)\\ \beta &\mapsto \epsilon \circ \beta{\uparrow}_H^G\end{aligned}$$

It is possible to show that these are mutually inverse, so $\Psi(\Phi(\psi)) = \psi$ for all $\psi \in \mathrm{Hom}(U{\uparrow}_H^G, V)$, $\Phi(\Psi(\beta)) = \beta$ for all $\beta \in \mathrm{Hom}_{KH}(U, V{\downarrow}_H^G)$ and

$$\mathrm{Hom}_{KG}(U{\uparrow}_H^G, V) \cong \mathrm{Hom}_{KH}(U, V{\downarrow}_H^G).$$

Moreover, these isomorphisms are *natural* in U and V which means in particular that for any KH-homomorphism $\gamma : U_1 \to U_2$, the following diagram commutes,

$$\begin{array}{ccc}\mathrm{Hom}_{KG}(U_1{\uparrow}_H^G, V) & \xrightarrow{\ \Phi\ } & \mathrm{Hom}_{KH}(U_1, V{\downarrow}_H^G)\\ {\scriptstyle \_\circ\gamma{\uparrow}_H^G}\big\uparrow & & \big\uparrow{\scriptstyle \_\circ\gamma}\\ \mathrm{Hom}_{KG}(U_2{\uparrow}_H^G, V) & \xrightarrow[\ \Phi\ ]{} & \mathrm{Hom}_{KH}(U_2, V{\downarrow}_H^G)\end{array}$$

and for any KG-homomorphism $\alpha : V_1 \to V_2$, the following diagram commutes.

$$\begin{array}{ccc}\mathrm{Hom}_{KH}(U, V_1{\downarrow}_H^G) & \xrightarrow{\ \Psi\ } & \mathrm{Hom}_{KG}(U{\uparrow}_H^G, V_1)\\ {\scriptstyle \alpha\circ\_}\big\downarrow & & \big\downarrow{\scriptstyle \alpha\circ\_}\\ \mathrm{Hom}_{KH}(U, V_2{\downarrow}_H^G) & \xrightarrow[\ \Psi\ ]{} & \mathrm{Hom}_{KG}(U{\uparrow}_H^G, V_2)\end{array}$$

Proposition 27.7 (*Characteristics of relative projectivity*)

Let $H \leqslant G$. Let U be a KG-module. Then the following are equivalent.

(a) The KG-module U is relatively H-projective.

(b) If $\psi : U \to W$ is a KG-homomorphism, $\varphi : V \twoheadrightarrow W$ is a surjective KG-homomorphism and there exists a KH-homomorphism $\alpha_H : U{\downarrow}_H^G \to V{\downarrow}_H^G$ such that $\varphi \circ \alpha_H = \psi$ on $U{\downarrow}_H^G$, then there exists a KG-homomorphism $\alpha_G : U \to V$ such that $\varphi \circ \alpha_G = \psi$ so that the diagram on the right commutes.

$$\begin{array}{ccc} & & U\\ & \swarrow{\scriptstyle \alpha_G} & \big\downarrow{\scriptstyle \psi}\\ V & \xrightarrow[\ \varphi\ ]{} & W\end{array}$$

(c) Whenever $\varphi : V \to U$ is a surjective KG-homomorphism such that the restriction $\varphi : V{\downarrow}_H^G \to U{\downarrow}_H^G$ is a split surjective KH-homomorphism, then φ is a split surjective KG-homomorhpism.

(d) The following surjective KG-homomorphism is split.

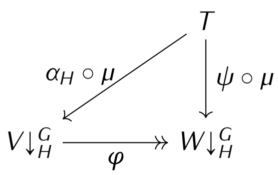
$$U{\downarrow}_H^G{\uparrow}_H^G = KG \otimes_{KH} U \to U$$
$$x \otimes u \mapsto xu$$

(e) The KG-module U is a direct summand of $U{\downarrow}_H^G{\uparrow}_H^G$.

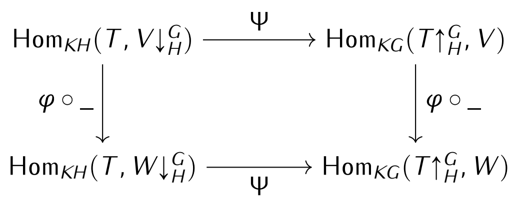
Proof:

(a)⇒(b): First we consider the case where $U = T{\uparrow}_H^G$ is an induced module. Suppose that we have KG-homomorphisms $\psi : T{\uparrow}_H^G \to W$ and $\varphi : V \twoheadrightarrow W$ as shown in the the diagram on the left. Suppose that there exists a KH-homomorhpism $\alpha_H : T{\uparrow}_H^G{\downarrow}_H^G \to V{\downarrow}_H^G$ such that $\psi = \varphi \circ \alpha_H$ on $T{\uparrow}_H^G{\downarrow}_H^G$, that is, the diagram on the right commutes.

Let $\mu : T \to T{\uparrow}_H^G{\downarrow}_H^G$ and $\epsilon : T{\downarrow}_H^G{\uparrow}_H^G \to T$ be as defined in Notation 27.6, so μ is an injective KH-homomorphism and ϵ is a surjective KG-homomorphism. Then the following triangle of KH-modules and KH-homomorphisms commutes.



By the naturality of Φ and Ψ from Notation 27.6, since $\varphi : V \to W$ is a KG-homomorphism, we have the following commutative diagram.



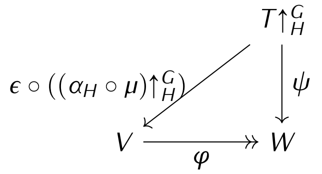
Hence the following two KG-homomorphisms $T{\uparrow}_H^G \to W$ are equal.

$$\Psi(\varphi \circ (\alpha_H \circ \mu)) = \varphi \circ (\Psi(\alpha_H \circ \mu))$$

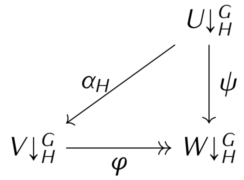
By the commutativity of the previous triangle, the left hand side of this equation is equal to $\Psi(\psi \circ \mu) = \Psi(\Phi(\psi)) = \psi$ since Ψ and Φ are inverse to one another. Thus

$$\psi = \varphi \circ \epsilon \circ ((\alpha_H \circ \mu){\uparrow}_H^G)$$

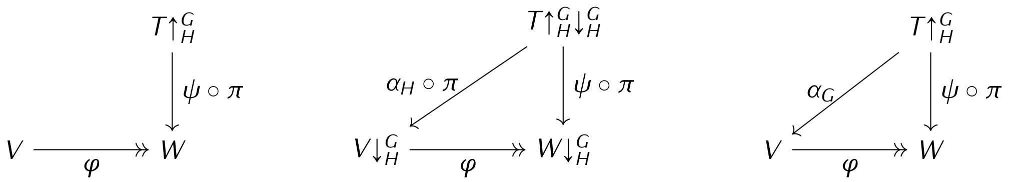
and so the following triangle of KG-homomorphisms commutes, proving the implication for $U = T{\uparrow}_H^G$ an induced module.



Now let U be any summand of $T\uparrow_H^G$. Let $U \xrightarrow{\iota} T\uparrow_H^G \xrightarrow{\pi} U$ denote the inclusion and projection maps. Suppose that there is a KH-homomorphism $\alpha_H : U\downarrow_H^G \to V\downarrow_H^G$ such that $\varphi \circ \alpha_H = \psi$ on $U\downarrow_H^G$.

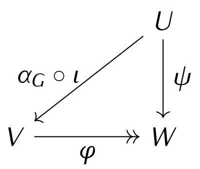


Then we have the following diagrams.

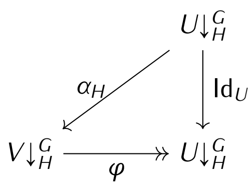


The first is a diagram of KG-homomorphisms. The middle diagram of KH-homomorphisms commutes by definition of α_H, and hence by the first part there is a KG-homomorphism $\alpha_G : T\uparrow_H^G \to V$ such that $\varphi \circ \alpha_G = \psi \circ \pi$, so the third diagram of KG-homomorhpisms also commutes.

Now $\varphi \circ \alpha_G \circ \iota = \psi \circ \pi \circ \iota = \psi$, so $\alpha_G \circ \iota : U \to V$ is a KG-module homomorphism such that $\varphi \circ (\alpha_G \circ \iota) = \psi$ and the following triangle commutes, as required.



(b)⇒(c): Let $\varphi : V \to U$ be a surjective KG-homomorphism which is split as a KH-homomorphism. Suppose that α_H is a KH-section for φ, so we have the following commutative diagram of KH-modules.



Then assuming (b) is true, there exists a KG-homomorphism $\alpha_G : U \to V$ such that $\varphi \circ \alpha_G = \mathrm{Id}_U$. In particular, α_G is a KG-section for φ, so $\varphi : V \to U$ is a split surjective KG-homomorphism.

(c)⇒(d): As mentioned in Notation 27.6, μ is a KH-section for ϵ, so $\epsilon : U\downarrow_H^G\uparrow_H^G \to U$ is split as a KH-homomorphism. Hence by part (c), $\epsilon : U\downarrow_H^G\uparrow_H^G \to U$ is split as a KG-homomorhpism, showing part (d).

(d)⇒(e): immediate

(e)$\Rightarrow$(a): immediate ■

Exercise 27.8 (*Relative projectivity*)

Let $H \leqslant G$. Let U be a KG-module. Prove that if U is H-projective and W is a KG-module, then $U \otimes_K W$ is H-projective.

In the next theorem we see a situation where we can always find relatively projective modules.

Theorem 27.9

Let $H \leqslant G$ such that H contains a Sylow p-subgroup of G. Then every KG-module is H-projective.

Before proving this theorem, let us consider its application to the case when $H = 1$.

Example 27.10

Let $H = 1$. If H contains a Sylow p-subgroup of G then the Sylow p-subgroups of G are trivial, so p does not divide the order of G. The theorem then shows that all KG-modules are 1-projective and hence projective. We know this already, however! If p does not divide the order of G then KG is semisimple (Maschke's Theorem 16.1), and so all KG-modules are projective by Chapter 6 Example 2 (d).

Proof: Let V be a KG-module and let $H \leqslant G$ such that H contains a Sylow p-subgroup of G. Let $\varphi : U \twoheadrightarrow V$ be a surjective KG-homomorphism which splits as a KH-homomorhpism. We will show that φ splits as a KG-homomorphism, and hence V satisfies Theorem 27.7 (c) so V is H-projective.

Then $U \cong W \oplus V$ as KH-modules, where $W := \ker(\varphi)$. Let $f : U \to W$ be a projection map onto the first factor. Note that since H contains a Sylow p-subgroup of G, $|G : H|$ is coprime to p. Thus $|G : H|$ is invertible in K because K is of characteristic p. We can therefore define a map $\tilde{f} : U \to W$ as follows:

$$\tilde{f}(u) := \frac{1}{|G:H|} \sum_{g \in [G/H]} g^{-1} f(gu) \qquad \text{for } u \in U,$$

where the sum runs over a set of left coset representatives of H in G. Since $gu \in U$ (U is a KG-module), and $f(gu) \in W$ by definition of f, $\tilde{f}(u) \in W$ so the map is well defined. Also, for any $g' \in G$,

$$\begin{aligned}
\tilde{f}(g'u) &= \frac{1}{|G:H|} \sum_{g \in [G/H]} g^{-1} f(gg'u) \\
&= \frac{1}{|G:H|} \sum_{g \in [G/H]} g'(gg')^{-1} f(gg'u) \\
&= \frac{1}{|G:H|} \sum_{g'' \in [G/H]} g'(g'')^{-1} f(g''u) \\
&= g' \frac{1}{|G:H|} \sum_{g'' \in [G/H]} g''^{-1} f(g''u) \\
&= g'\tilde{f}(u).
\end{aligned}$$

Thus, $\tilde{f} : U \to W$ is in fact a KG-homomorphism.

Now, for any $w \in W$ we have

$$\tilde{f}(w) = \frac{1}{|G:H|} \sum_{g \in [G/H]} g^{-1} f(gw) = \frac{1}{|G:H|} \sum_{g \in [G/H]} g^{-1} gw = \frac{1}{|G:H|} \sum_{g \in [G/H]} w = w,$$

which shows that $\ker(\tilde{f}) \cap W = \{0\}$ and $\tilde{f}^2 = \tilde{f}$.

Finally, for any $u \in U$ we have $u = (u - \tilde{f}(u)) + \tilde{f}(u)$, so

$$\tilde{f}(u) = \tilde{f}(u - \tilde{f}(u)) + \tilde{f}^2(u) = \tilde{f}(u - \tilde{f}(u)) + \tilde{f}(u).$$

In particular, $\tilde{f}(u - \tilde{f}(u)) = 0$ and $u - \tilde{f}(u) \in \ker(\tilde{f})$. Hence every element of U can be expressed as the sum of an element in $\ker(\tilde{f})$ and an element of W, so $U \cong W \oplus \ker \tilde{f}$ as KG-modules. Hence φ splits as a KG-homomorphism, so V is H-projective by Theorem 27.7. ■

Corollary 27.11

Suppose that a subgroup H of G contains a Sylow p-subgroup of G. Then a KG-module U is projective if and only if $U\downarrow_H^G$ is projective.

Proof:

$\Rightarrow$: Lemma 26.3.

$\Leftarrow$: Suppose that $U\downarrow_H^G$ is projective and H contains a Sylow p-subgroup of G. Then $U\downarrow_H^G$ is a summand of a free module $(KH^\circ)^n$, and every KG-module is H-projective. In particular, U is H-projective so U is a summand of $U\downarrow_H^G\uparrow_H^G$ by Theorem 27.7. Hence U is a summand of $U\downarrow_H^G\uparrow_H^G$ which is a summand of $(KH^\circ)^n\uparrow_H^G = (KG^\circ)^n$, so U is projective. ■

28 Vertices and Sources

Theorem 28.1

Let U be an indecomposable KG-module.

(a) There is a unique conjugacy class of subgroups Q of G that are minimal subject to the property that U is Q-projective.

(b) Let Q be a minimal subgroup of G such that U is Q-projective. There is an indecomposable KQ-module T that is unique up to conjugacy by elements of $N_G(Q)$ such that U is a direct summand of $T\uparrow_Q^G$. Such a T is necessarily a direct summand of $U\downarrow_Q^G$.

Proof:

(a) Suppose that U is both H- and K-projective for subgroups H and K of G. Then U is a direct summand of $U\downarrow_H^G\uparrow_H^G$ and $U\downarrow_K^G\uparrow_K^G$ by Proposition 27.7 (e). Hence U is also a direct summand of $U\downarrow_H^G\uparrow_H^G\downarrow_K^G\uparrow_K^G$. By the Mackey formula and transitivity of induction and restriction, it follows that

$$\begin{aligned} U\downarrow_H^G\uparrow_H^G\downarrow_K^G\uparrow_K^G &= \left(\left(U\downarrow_H^G\right)\uparrow_H^G\downarrow_K^G\right)\uparrow_K^G \\ &= \left(\bigoplus_{g\in[K\backslash G/H]} \left({}^g\left(U\downarrow_H^G\right)\downarrow^{{}^gH}_{K\cap {}^gH}\right)\uparrow^K_{K\cap {}^gH}\right)\uparrow_K^G \\ &= \bigoplus_{g\in[K\backslash G/H]} \left({}^gU\downarrow^G_{K\cap {}^gH}\right)\uparrow^G_{K\cap {}^gH}. \end{aligned}$$

Therefore U is a direct summand of some module induced from $K \cap {}^gH$ for some $g \in G$. In other words, U is relatively $K \cap {}^gH$-projective. Suppose that both K and H are minimal such that U is projective with respect to these groups. Then $K \cap {}^gH = K$ so $K \subseteq {}^gH$ and $H \subseteq {}^{g^{-1}}K$, hence H and K are G-conjugate.

(b) Let Q be a minimal subgroup relative to which U is projective. Then U is a direct summand of $U{\downarrow}^G_Q{\uparrow}^G_Q$ so it is a direct summand of $T{\uparrow}^G_Q$ for some indecomposable direct summand T of $U{\downarrow}^G_Q$. If T' is another indecomposable KQ-module such that U is a direct summand of $T'{\uparrow}^G_Q$, then T is a direct summand of $T'{\uparrow}^G_Q{\downarrow}^G_Q$. Mackey's formula says that

$$T'{\uparrow}^G_Q{\downarrow}^G_Q = \bigoplus_{g\in[Q\backslash G/Q]} ({}^gT'{\downarrow}^{{}^gQ}_{Q\cap {}^gQ}){\uparrow}^Q_{Q\cap {}^gQ}\,.$$

Hence T is a direct summand of $({}^gT'{\downarrow}^{{}^gQ}_{Q\cap {}^gQ}){\uparrow}^Q_{Q\cap {}^gQ}$, and therefore U is relatively $Q\cap {}^gQ$-projective, for some $g \in G$. Since Q is a minimal subgroup relative to which U is projective, $Q = Q\cap {}^gQ$ and hence $g \in N_G(Q)$. It follows that T is actually a direct summand of ${}^gT'$, for this $g \in G$. Since T and T' are indecomposable, however, this means that $T = {}^gT'$, so T is unique up to conjugacy by elements of $N_G(Q)$.

Now $T = {}^gT'$ is an idecomposable direct summand of $U{\downarrow}^G_Q$ by definition, so $T' = {}^{g^{-1}}T$ is a direct summand of $({}^{g^{-1}}U){\downarrow}^G_Q$. However, $U \cong {}^{g^{-1}}U$ as KG-modules, so this means that T' is also a direct summand of $U{\downarrow}^G_Q$. ■

Definition 28.2

Let U be an indecomposable kG-module. A **vertex** of U is a minimal subgroup Q of G such that U is relatively Q-projective. The vertices of U are unique up to G-conjugacy.
A **KQ-source**, or simply **source** of U is a KQ-module T for which U is a direct summand of $T{\uparrow}^G_Q$, for some vertex Q of U. For a fixed vertex Q, the sources of U are unique up to $N_G(Q)$-conjugacy.

Exercise 28.3

Let $H \leqslant G$ and $J \leqslant G$. Let U be a KG-module. If U is H-projective and W is an indecomposable direct summand of $U{\downarrow}^G_J$ then W is $J\cap {}^gH$-projective for some element $g \in G$, and there is a vertex of W that is contained in this subgroup $J \cap {}^gH$.

The idea is that the closer the vertex of a module is to the trivial group, the closer the module is to being projective: a KG-module U with trivial vertex is $\{1\}$-projective and hence projective.

Proposition 28.4

(a) The vertices of an indecomposable KG-module are p-groups.

(b) If P is a p-group and H is a subgroup of P then $K{\uparrow}^P_H$ is an indecomposable KP-module.

(c) The vertices of the trivial KG-module K are Sylow p-subgroups of G.

Proof:

(a) By Theorem 27.9, we know that every KG-module is projective relative to a Sylow p-subgroup of G. Therefore vertices are contained in Sylow p-subgroups, and hence are themselves p-groups.

(b) Because P is a p-group, the only simple KP-module is the trivial module K (see Cor. 17.3). Moreover,

$$\begin{aligned}\dim \operatorname{soc}(K{\uparrow}^P_H) &= \dim \operatorname{Hom}_{KP}(K, K{\uparrow}^P_H)\\ &= \dim \operatorname{Hom}_{KH}(K{\downarrow}^P_H, K)\end{aligned}$$

by Frobenius reciprocity (Theorem 20.10 (b)). But $\operatorname{Hom}_{KH}(K{\downarrow}^P_H, K) \cong K$ so this, and hence $\operatorname{soc}(K{\uparrow}^P_H)$, has dimension 1. If $K{\downarrow}^P_H$ is decomposable then $K{\downarrow}^P_H = U \oplus V$ for some KP-modules

U and V, and hence $\mathrm{soc}(K\downarrow_H^P) = \mathrm{soc}(U) \oplus \mathrm{soc}(V)$. This contradicts the fact that $\mathrm{soc}(K\uparrow_H^P)$ has dimension 1, therefore $K\uparrow_H^P$ is indecomposable.

(c) Let Q be a vertex of K and let P be a Sylow p-subgroup of G which contains Q. Then $K \mid K\uparrow_Q^G$, so $K\downarrow_P^G$ is a summand of $K\uparrow_Q^G\downarrow_P^G = \bigoplus_{g\in[P\backslash G/Q]} K\uparrow_{P\cap {}^gQ}^P$ and hence is a summand of $K\uparrow_{P\cap {}^gQ}^P$ for some $g \in G$. By part (b), since P is a p-group, $K\uparrow_{P\cap {}^gQ}^P$ is indecomposable. Thus $K\downarrow_P^G = K\uparrow_{P\cap {}^gQ}^P$ and hence $P \cap {}^gQ = P$, so Q is a Sylow p-subgroup of G. ■

29 The Green Correspondence

The Green correspondence is used to reduce questions about indecomposable modules to a situation where the vertex of the module is a normal subgroup. This technique is very useful in many situations, particularly in block theory. Many properties in modular representation theory are believed to be determined by normalisers of p-subgroups.

We will need the following easy properties of vertices and sources in the proof of Green's correspondence.

Exercise 29.1

Prove the following Lemma.

Lemma 29.2

Let Q be a p-subgroup of G and let L be a subgroup of G containing $N_G(Q)$.

(a) Suppose that V is an indecomposable KL-module with vertex Q and let U be a direct summand of $V\uparrow_L^G$ such that V is a direct summand of $U\downarrow_L^G$. Then Q is also a vertex of U.

(b) Suppose that V is an indecomposable KL-module which is Q-projective and there exists an indecomposable direct summand U of $V\uparrow_L^G$ with vertex Q. Then V also has vertex Q.

Exercise 29.3

Let U be an indecomposable KG-module with vertex Q and KQ-source S and let L be a subgroup of G containing Q. Prove that there exists an indecomposable direct summand of $U\downarrow_L^G$ with vertex Q.

Theorem 29.4 (*Green Correspondence*)

Let Q be a p-subgroup of G and let L be a subgroup of G containing $N_G(Q)$.

(a) Let U be an indecomposable KG-module with vertex Q. Then in any decomposition of $U\downarrow_L^G$ into a direct sum of indecomposable modules, there is a unique indecomposable direct summand with vertex Q which we denote by $f(U)$. Writing $U\downarrow_L^G = f(U) \oplus X$, then every direct summand of X is projective relative to a subgroup of the form $L \cap {}^xQ$ for some $x \in G\backslash L$.

(b) Let V be an indecomposable KL-module with vertex Q. Then in any decomposition of $V\uparrow_L^G$ into a direct sum of indecomposable modules, there is a unique indecomposable direct summand with vertex Q which we denote by $g(V)$. Writing $V\uparrow_L^G = g(V) \oplus Y$, then every direct summand of Y is projective relative to a subgroup of the form $Q \cap {}^xQ$ for some $x \in G\backslash L$.

(c) With this notation, we then have $g(f(U)) \cong U$ and $f(g(V)) \cong V$.

Proof: We first note some properties of the groups $Q \cap {}^xQ$ and $L \cap {}^xQ$ for $x \in G \backslash L$.

- Since $N_G(Q) \leqslant L$, x does not normalize Q and hence $Q \cap {}^xQ$ is a proper subgroup of Q.
- $L \cap {}^xQ$ may be the same size of Q, in which case it is equal to ${}^xQ$.
- Suppose that $L \cap {}^xQ$ is conjugate to Q in L: $L \cap {}^xQ = {}^zQ$ for some $z \in L$. Then ${}^xQ = {}^zQ$ so ${}^{z^{-1}x}Q = Q$ and hence $z^{-1}x \in N_G(Q) \leqslant L$. Therefore $x \in zL = L$. This contradicts $x \in G \backslash L$. Therefore $L \cap {}^xQ$ is not conjugate to Q in L.

Let V be an indecomposable KL-module with vertex Q.

Claim: Any decomposition of $V\uparrow_L^G\downarrow_L^G$ into a direct sum of indecomposable KL-modules has a unique direct summand with vertex Q, and all other direct summands are projective relative to subgroups of the form $L \cap {}^xQ$ with $x \notin L$.

Pf of claim Let T be a KQ-source for V and write $T\uparrow_Q^L = V \oplus Z$ for some KL-module Z. Let V' and Z' denote KL-modules such that $V\uparrow_L^G\downarrow_L^G = V \oplus V'$ and $Z\uparrow_L^G\downarrow_L^G = Z \oplus Z'$. Then, on the one hand we have

$$\begin{aligned} T\uparrow_Q^G\downarrow_L^G &\cong (V \oplus Z)\uparrow_L^G\downarrow_L^G \\ &= V\uparrow_L^G\downarrow_L^G \oplus Z\uparrow_L^G\downarrow_L^G \\ &= V \oplus V' \oplus Z \oplus Z'. \end{aligned}$$

On the other hand, by Mackey we also have

$$\begin{aligned} T\uparrow_Q^G\downarrow_L^G &= \bigoplus_{x\in[L\backslash G/Q]} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ} \\ &\cong T\uparrow_Q^L \bigoplus_{x\in[L\backslash G/Q]\backslash L} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ} \\ &= V \oplus Z \bigoplus_{x\in[L\backslash G/Q]\backslash L} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ}. \end{aligned}$$

Therefore

$$V \oplus V' \oplus Z \oplus Z' \cong V \oplus Z \bigoplus_{x\in[L\backslash G/Q]\backslash L} ({}^xT\downarrow^{{}^xQ}_{L\cap {}^xQ})\uparrow^L_{L\cap {}^xQ}.$$

Clearly all direct summands not in $V \oplus Z$ are projective relative to subgroups of the form $L \cap {}^xQ$ for some $x \notin L$. We already saw that $L \cap {}^xQ$ is not conjugate to Q for any $x \notin L$. Hence V is the unique direct summand of $V\uparrow_L^G\downarrow_L^G = V \oplus V'$ with vertex Q, and all other direct summands in V' are projective relative to subgroups of the form $L \cap {}^xQ$ with $x \notin L$.

Pf of (b) We continue with the notation above, with V an indecomposable KL-module with vertex Q. Write $V\uparrow_L^G$ as a direct sum of indecomposable KG-modules and pick a direct summand U such that $U\downarrow_L^G$ has V as a direct summand. By Lemma 29.2 (a), since Q is a vertex of V, Q is also a vertex of U. Therefore $V\uparrow_L^G$ has at least one direct summand with vertex Q.

Let U' be another direct summand of $V\uparrow_L^G$. Then $V\uparrow_L^G = U \oplus U' \oplus X$ for some KG-module X, so in the notation of the claim, $V \oplus V' = U\downarrow_L^G \oplus U'\downarrow_L^G \oplus X\downarrow_L^G$. Therefore $U'\downarrow_L^G$ is a direct summand of V' and hence every indecomposable direct summand of $U'\downarrow_L^G$ is projective relative to a subgroup $L \cap {}^yQ$, for some $y \notin L$. Now since V is a direct summand of $T\uparrow_Q^L$ and U' is a direct summand of $V\uparrow_L^G$, it follows that U' is a direct summand of $T\uparrow_Q^G$ and hence U' is projective relative to Q. Hence U' has a vertex Q' which is a subgroup of Q.

Let S be a KQ'-source of U'. Theorem 28.1 (b) shows that S is a direct summand of $U'\downarrow_{Q'}^G$. Since $Q' \leqslant L$, $U'\downarrow_{Q'}^G = U'\downarrow_L^G\downarrow_{Q'}^L$ and hence S is a direct summand of $Y\downarrow_{Q'}^L$ for some indecomposable direct summand Y of $U'\downarrow_L^G$. It follows from Exercise 29.3 that Q' is also a vertex of Y. But the indecomposable direct summands of $U'\downarrow_L^G$ are projective relative to subgroups of the form $L \cap {}^yQ$ for some $y \notin L$. Therefore one of the subgroups $L \cap {}^yQ$ with $y \notin L$ contains an L-conjugate of Q' – in other words, ${}^zQ' \subseteq L \cap {}^yQ$ for some $z \in L$. Hence $Q' \subseteq {}^{z^{-1}y}Q$ where $z^{-1}y \notin L$. This shows that $Q' \subseteq Q \cap {}^xQ$ for some $x \notin L$, proving part (b) with $g(V) := U$.

Pf of (a) Suppose now that U is an indecomposable KG-module with vertex Q and let T be a KQ-source of U. Then U is a direct summand of $T\uparrow_Q^G = T\uparrow_Q^L\uparrow_L^G$, so there is an indecomposable direct summand V of $T\uparrow_Q^L$ such that U is a direct summand of $V\uparrow_L^G$. This means that V is Q-projective (since it is a direct summand of $T\uparrow_Q^L$), and so by Lemma 29.2 (b), Q is a vertex of V.

By Exercise 29.3, there exists an indecomposable direct summand Y of $U\downarrow_L^G$ with vertex Q. But $U\downarrow_L^G$ is a direct summand of $V\uparrow_L^G\downarrow_L^G$ and the claim shows that the only direct summand of $V\uparrow_L^G\downarrow_L^G$ with vertex Q is V. Therefore $Y \cong V$ and in any expression of $U\downarrow_L^G$ as a direct sum of indecomposables, one direct summand is isomorphic to V and the rest are projective relative to subgroups of the form $L \cap {}^xQ$ for some $x \notin L$. This proves part (a).

Pf of (c) Finally, part (c) follows from parts (a) and (b) and the fact that U is isomorphic to a direct summand of $U\downarrow_L^G\uparrow_L^G$ and V is isomorphic to a direct summand of $V\uparrow_L^G\downarrow_L^G$. ■